Introduction to Logarithms

Victor I. Piercey

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In my senior thesis, I wanted to estimate productivity in the Russian defense sector in the mid-1990’s.

In particular I wanted to test for “Cobb-Douglas” production technology:

$$Q = AL^{\beta_1} K^{\beta_2}$$

where $Q$ is output, $L$ is the amount of labor input, $K$ is the amount of capital input, and $A, \beta_1, \beta_2$ are productivity parameters.

I had data for $Q, L$ and $K$ and wanted to run regression.

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Finding Inverse Functions

If your function is given by a formula \( y = f(x) \), the formula for the inverse function is accomplished by switching \( x \) and \( y \) (hence switching inputs and outputs) and solving for \( y \).

For example: find the inverse of the function \( f(x) = 2x + 8 \).
If your function is given by a formula $y = f(x)$, the formula for the inverse function is accomplished by switching $x$ and $y$ (hence switching inputs and outputs) and solving for $y$.

For example: find the inverse of the function $f(x) = 2x + 8$. 

The equation $y = 2^x$ defines a function. We can plug in values and graph this function if we like.

More generally, if $0 < b < 1$ or $b > 1$, then the equation $y = b^x$ defines a function, called the exponential function of base $b$.

The most common bases are $b = e$ and $b = 10$.

Try and find the inverse function for $y = 2^x$. 
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Try and find the inverse function for \( y = 2^x \).
There is an inverse function, but ordinary algebra will not help find it.

Instead, we invent notation to define the inverse function:

\[ y = \log_2(x). \]

By definition, \( \log_b(x) = y \) means \( b^y = x \).

This is the logarithm with base \( b \).
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Practice Computing Some Logarithms

For practice, let’s compute the following:

1. \( \log_2(4) \)
2. \( \log_3(27) \)
3. \( \log_7 \left( \frac{1}{7} \right) \)
4. \( \log_b(1) \) for any base \( b \)
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The base $b = e$ occurs frequently in nature, so the logarithm with base $e$ is called the natural log and it is denoted $\ln(x)$.

The base $b = 10$ is very common, so it is called the common log and is denoted $\log(x)$, with the base suppressed.

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These are the only logarithms that can be computed on your calculator (without using the change of base formula).
The reason logarithms are so important is because of their properties, which come from properties of exponents.

The following are the properties of exponents:

1. $b^{n+m} = b^n b^m$
2. $b^{n-m} = \frac{b^n}{b^m}$
3. $(b^n)^m = b^{nm}$.
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Properties of Logarithms

Since logarithms are inverses of exponential functions, one can invert the properties of exponents to obtain the following properties:

1. \( \log_b(MN) = \log_b(M) + \log_b(N) \)

2. \( \log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N) \)

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Use the properties of logarithms to expand the following:

1. \( \log_2(xy^2) \)

2. \( \ln \left( \frac{x^2y^3}{\sqrt{z}} \right) \)

3. \( \log \left( \frac{1}{x^6} \right) \).

Use the properties of logarithms to write the following as a single logarithm with no coefficient:

1. \( \log_3(x) - 2\log_3(y) \)

2. \( 4\ln(x) + 2\ln(y) \)

3. \( \frac{1}{2} \log(x) - \frac{1}{2} \log(y) \).
Recall that I wanted to run a regression with the non-linear model:

\[ Q = AL^{\beta_1} K^{\beta_2}. \]

To make this linear, I hit both sides with a logarithm and use the properties to obtain:

\[ \ln(Q) = \ln(A) + \beta_1 \ln(K) + \beta_2 \ln(L). \]

This is now a linear model (in two input variables) with intercept \( \ln(A) \) and slopes \( \beta_1 \) and \( \beta_2 \). I can now run regression.

Due to their properties, logarithms allow us to make linear models out of non-linear models.
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