The Squeeze Theorem: Statement and Example

1 The Statement

First, we recall the following "obvious" fact that limits preserve inequalities.

Lemma 1.1. *Suppose we have an inequality of functions*

$$g(x) \leq f(x) \leq h(x)$$

in an interval around c. Then

$$\lim_{x \to c} g(x) \leq \lim_{x \to c} f(x) \leq \lim_{x \to c} h(x)$$

provided those limits exist.

When the limits on the upper bound and lower bound are the same, then the function in the middle is "squeezed" into having the same limit. See page 61 in the text for more details.

Theorem 1.2 (The Squeeze Theorem)**.** *Suppose we have an inequality of functions*

$$g(x) \leq f(x) \leq h(x)$$

in a interval around c and that

$$\lim_{x \to c} g(x) = L = \lim_{x \to c} h(x).$$

Then

$$\lim_{x \to c} f(x) = L.$$

2 The Example

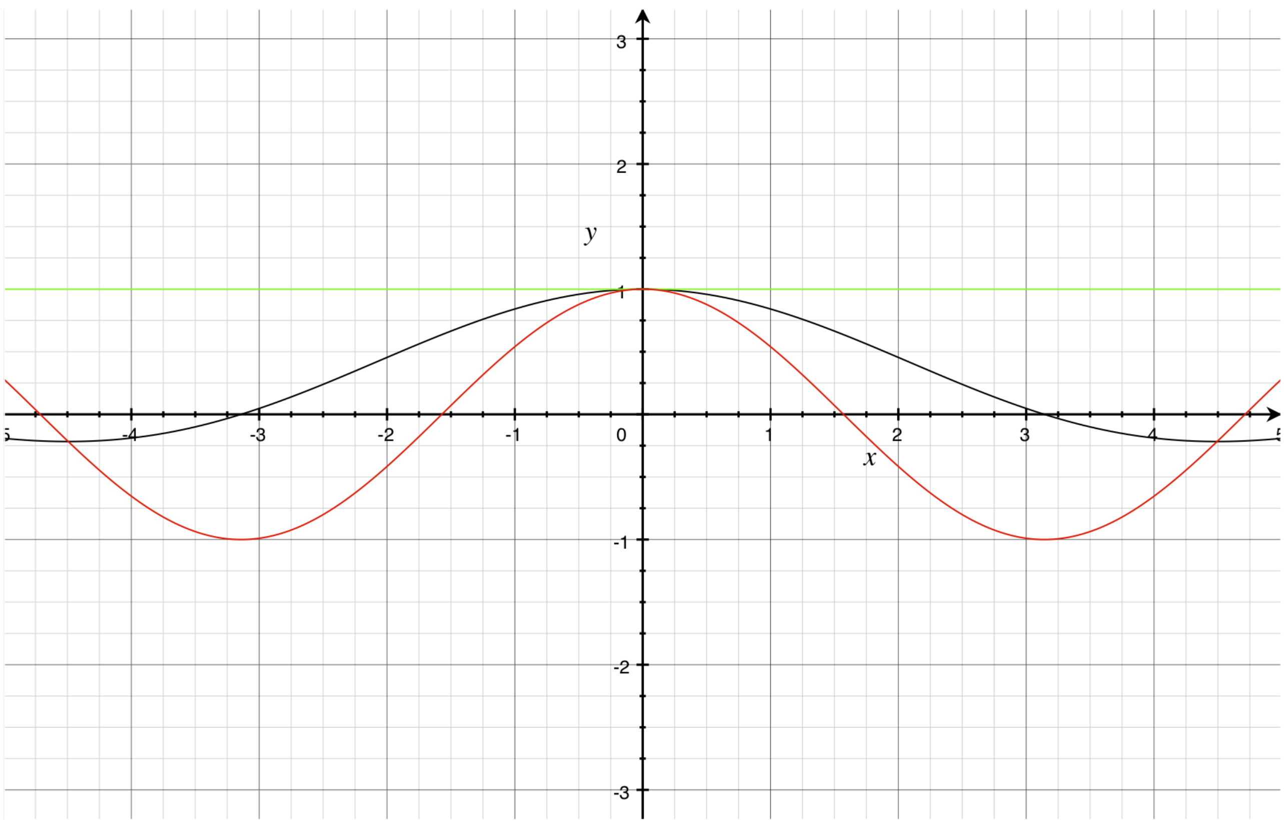
Our goal here is to show that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$

To do this, we'll use the Squeeze theorem by establishing upper and lower bounds on $\sin(x)/x$ in an interval around 0. Specifically, we'll show that

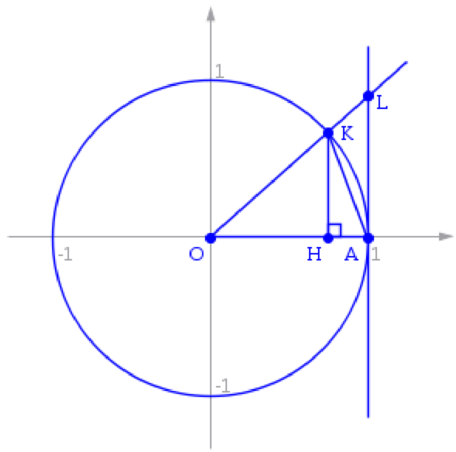
$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1$$

in an interval around 0. We can already see why this should be the case by the following graph.



$y = \cos(x)$ $\qquad$ $y = \dfrac{\sin(x)}{x}$ $\qquad$ $y = 1$

Note the symmetry in the graph: all of these functions are even functions (i.e., symmetric about the y-axis). Thus it suffices to establish these bounds on the interval $(0, \pi/2)$. With that said, let x be an acute angle and consider the following diagram on the unit circle where the angle $\angle KOH$ is the angle x.



We have

$$\begin{array}{ccccc}
\text{area of triangle } \triangle KOA & \leq & \text{area of sector } \triangleleft KOA & \leq & \text{area of triangle } \triangle LOA \\
\| & & \| & & \| \\
\dfrac{\sin(x)}{2} & \leq & \dfrac{x}{2} & \leq & \dfrac{\tan(x)}{2}
\end{array}$$

Hence multiplying through by $2/\sin(x) > 0$ gives

$$1 \leq \frac{x}{\sin(x)} \leq \frac{1}{\cos(x)},$$

so taking reciprocals yields

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1,$$

as needed.