TILING RECTANGLES WITH SQUARES
(FOR DUMMIES)

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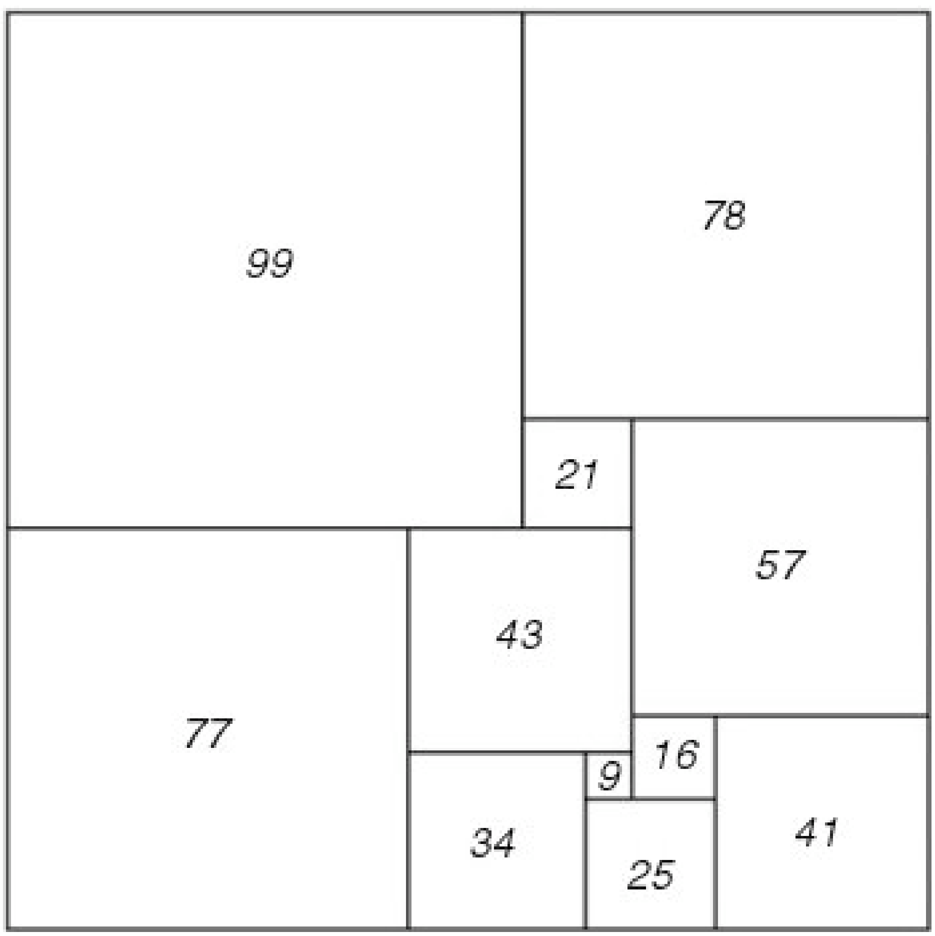
“Whence we came and whither we go be riddles,
and albeit such as these
we may never bring within our understanding,
yet there be many others
with which we and they that do come after us
will ever strive for the answer.
Whether success do attend or do not
attend our labour it is well that we make the attempt,
for 'tis truly good and honourable to train the mind,
and the wit and the fancy of man,
for out of such doth issue all manner of good
in ways unforeseen for them that do come after us."
Sir Hugh de Fortibus, from The Canterbury Puzzles
By Henry Ernest Dudeney

Tiling Rectangles with Squares: A Brief History

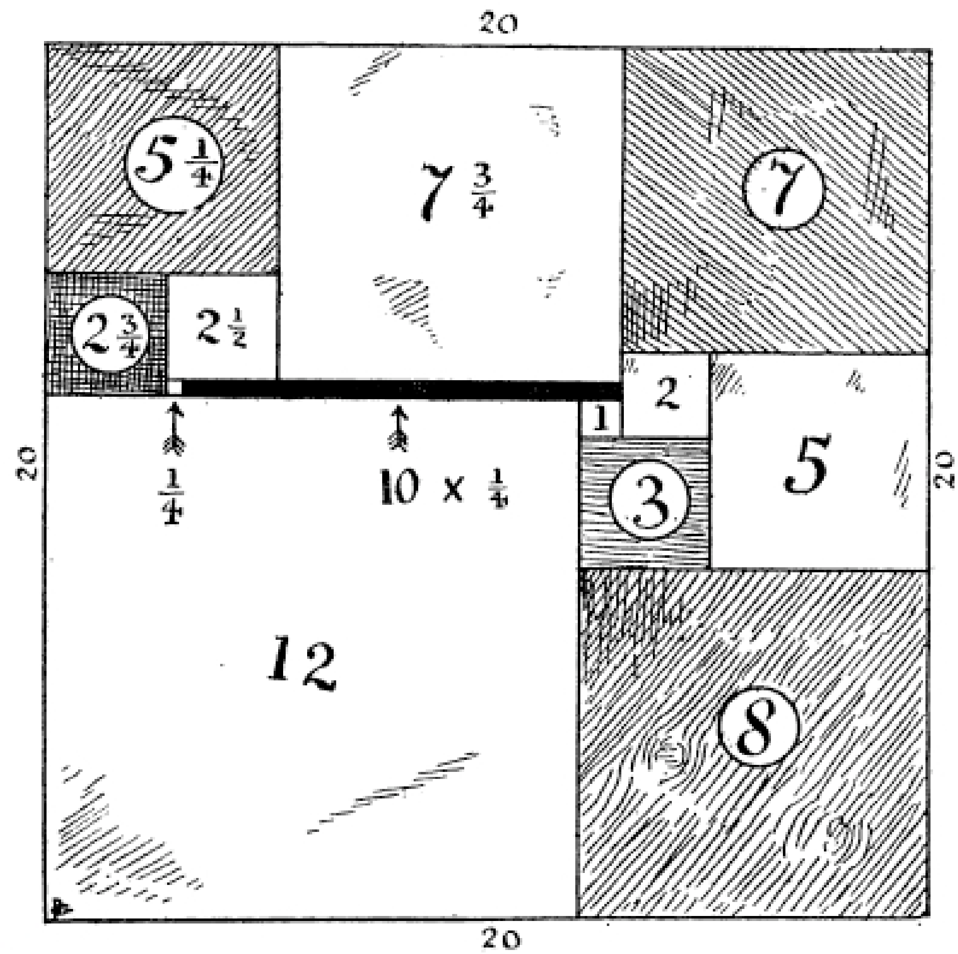
From at least the 20^{th} century and onward, mathematicians and puzzle enthusiasts have been interested in the notion of "tiling" plane figures. Some examples of tilings include tessellations, Penrose tilings, and real-life tilings such as ceilings and floors. The formal definition of "tiling" is "a plane-filling arrangement of plane figures or its generalization to higher dimensions" (Weisstein). In this paper, the tilings of rectangles, including squares, will be discussed. To tile a rectangle in this sense is to divide it up into smaller rectangles or squares. Each of the smaller rectangles or squares is called a tile. The side length of the smaller rectangle or square is called the size of the tile, and the number of different sizes of tiles determines the order of the tiling. For example, a rectangle that has been divided into 11 squares, all of which are different sizes, would be called a tiled (or squared) rectangle of order 11. The goal is to tile the rectangle such that none of the tiles overlap. *A rectangle that has been tiled with squares is "perfect" if none of the tiles is exactly the same size.* It is this problem that is the focus of this paper.



Above is an example of a perfect squared rectangle of order 11. None of the tiles overlap, and each is a different size. The numbers in each square represent the side lengths of the squares. The sides of the rectangle itself measure 176 and 177.

Upon initial examination of the problem, it would seem to the average puzzle enthusiast or mathematically inclined person that this is not such a difficult problem. One could surely use a guess-and-check method, and before long come up with a combination of numbers that works. Or, surely there is a formula that can be used to determine the dimensions of such a rectangle, and then the sizes of the inside squares can be fit into that solution. When one tries either of these methods, or any of many, many other methods that have been attempted, it quickly becomes clear that this problem is not what it seems!

The origins of this problem are somewhat of a mystery. A professor of mathematics and puzzle enthusiast named David Singmaster has researched the history of this problem extensively. According to Singmaster, the story of tiling rectangles with squares begins in 1902 with a puzzle titled "Lady Isabel's Casket," written by Henry Ernest Dudeney and published in the Perplexities column in Strand Magazine (squaring.net.history). The problem reappeared in Dudeney's book, "The Canterbury Puzzles," in 1910. The story of Lady Isabel's Casket tells of a young woman in the days of Chaucer, who was governed by a Sir Hugh de Fortibus. Isabel owned a box (referred to in the story as a casket), the top of which was made of a strip of gold and a certain number of inlaid wooden squares. When suitors asked for Isabel's hand in marriage, Sir Hugh promised to grant the request of that gentleman who could solve the following puzzle: Find "the dimensions of the top of the box from these facts alone: that there was a rectangular strip of gold, ten inches by ¼ inch; and the rest of the surface was exactly inlaid with pieces of wood, each piece being a perfect square, and no two pieces of the same size (Canterbury Puzzles, page #)." Dudeney published the following solution to the puzzle:



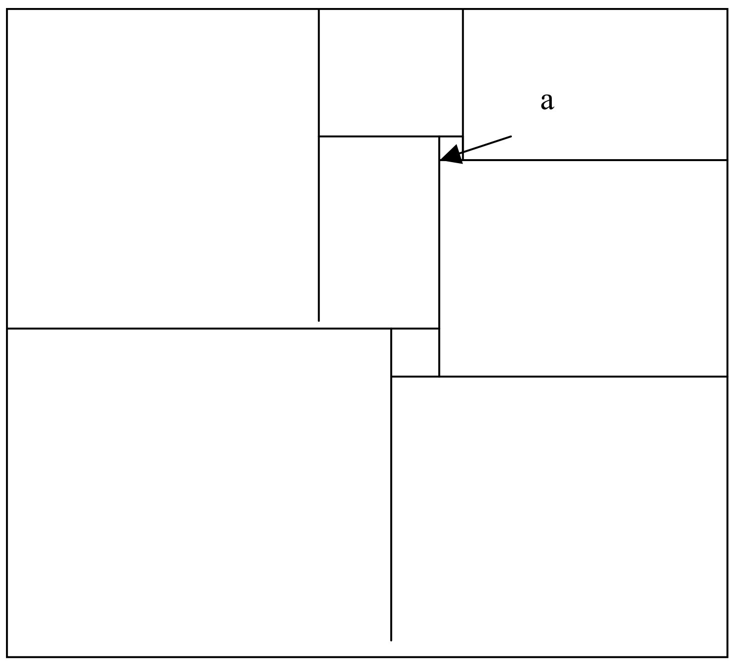
Dudeney asserts that this solution is unique, but he declines to explain his arrival at this conclusion: “This is the only possible solution, and it is a singular fact (though I cannot here show the subtle method of working) that the number, sizes, and order of those squares are determined by the given dimensions of the strip of gold, and the casket can have no other dimensions than 20 inches square (Canterbury Puzzles, 191).” Here we have nearly a squared square. That one 10 x ¼ rectangle contained in the solution may or may not have provided the spark of interest that led mathematicians to study this problem.

There are multiple solution methods for this problem. Three of those solution methods will be discussed. The first is a simple algebraic method that was used to catalogue squared rectangles of various sizes in search of a perfect squared square. The second is an electrical circuit graphing method that was used to make many interesting discoveries about what sorts of things are possible when tackling this problem, and led to rigorous proofs of that nature. Lastly, there are computer algorithms that search for new possible solutions.

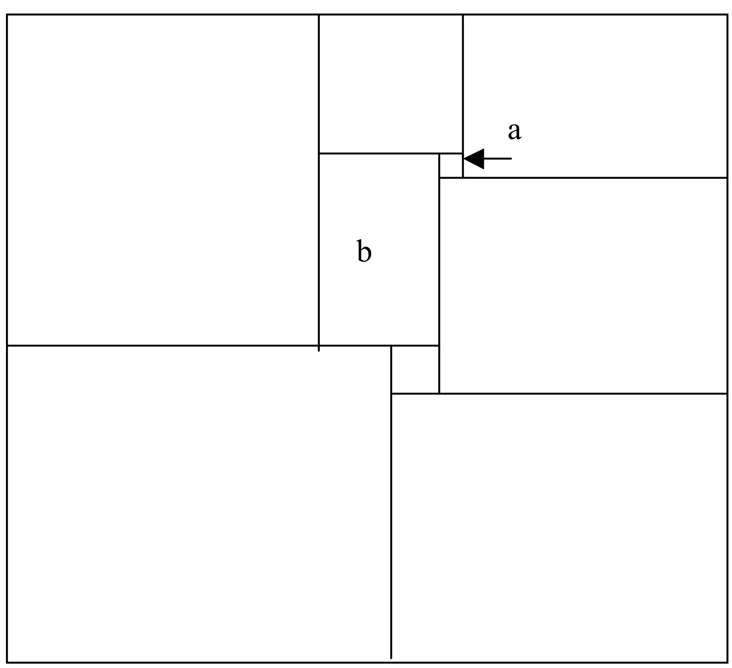
The Algebraic Solution

In 1903, only one year after Lady Isabel's Casket reportedly appeared in Strand Magazine, a mathematician named Max Dehn published a paper on the tiling of rectangles by rectangles in which he proved that in a perfect tiling, the sides of the large rectangle have to be integral multiples of each other. In other words, the sides have to be commensurable - capable of being measured with the same units, using counting numbers. Therefore, solutions should be presented with integer lengths (Squaring.net/Dehn). The fact that this paper was published only one year after Dudeney's puzzle seems to indicate that the problem of tiling rectangles and squares was being pondered in the years prior to Lady Isabel's Casket. Alternatively, Jasper Dale Skinner's research indicates that the problem originated with mathematicians from the University of Krakow sometime prior to or in the 1920's. No literature on the problem has been found dating earlier than 1902, but that does not exclude the possibility that it exists.

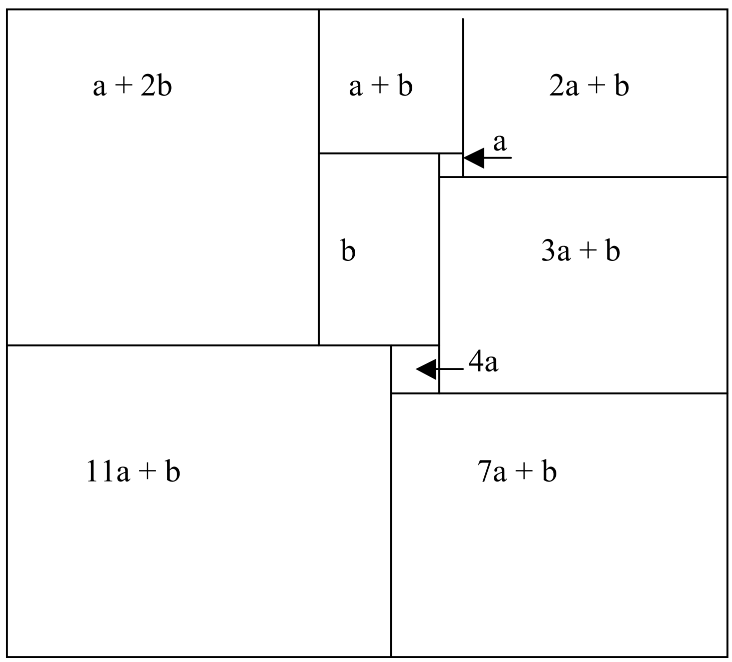
In 1925, Zbigniew Moron´ published the first perfect squared rectangle (Skinner, 10). Although this researcher could not identify the method with which Moron´ made his solutions, he may have used the algebraic method first employed by four Trinity College students who will be discussed later. To use the algebraic method of solution, we simply begin sketching squares into rectangles and substituting variables for the side lengths of the squares. Using variables allow us to name the smallest two or three elements as we see fit, and then find the side lengths of the other squares using those elements. This method is beautiful in its simplicity. Start with a sketch of a squared rectangle. It is not important that the sketch be to scale. We will use algebra and pretend each tile is a square. Choose a variable to represent the side length of the smallest square:



When we choose the second variable, we want to be able to build the measurements of the remaining side lengths using nothing but these two variables. If we choose b very carefully, we can accomplish this:



Now, we can use a and b to represent the side lengths of the remaining squares:



We can determine the side lengths of the large rectangle in terms of the variables. The upper horizontal side measures 4a + 4b. The lower horizontal side measures 18a + 2b. Since the sides must be equal, we can solve one of the variables for the other and then substitute a value for one to find a solution for the side measures of the rectangle.

18a + 2b = 4a + 4b

+ -2b + -2b

18a = 4a + 2b

+-4a +-4a

14a = 2b

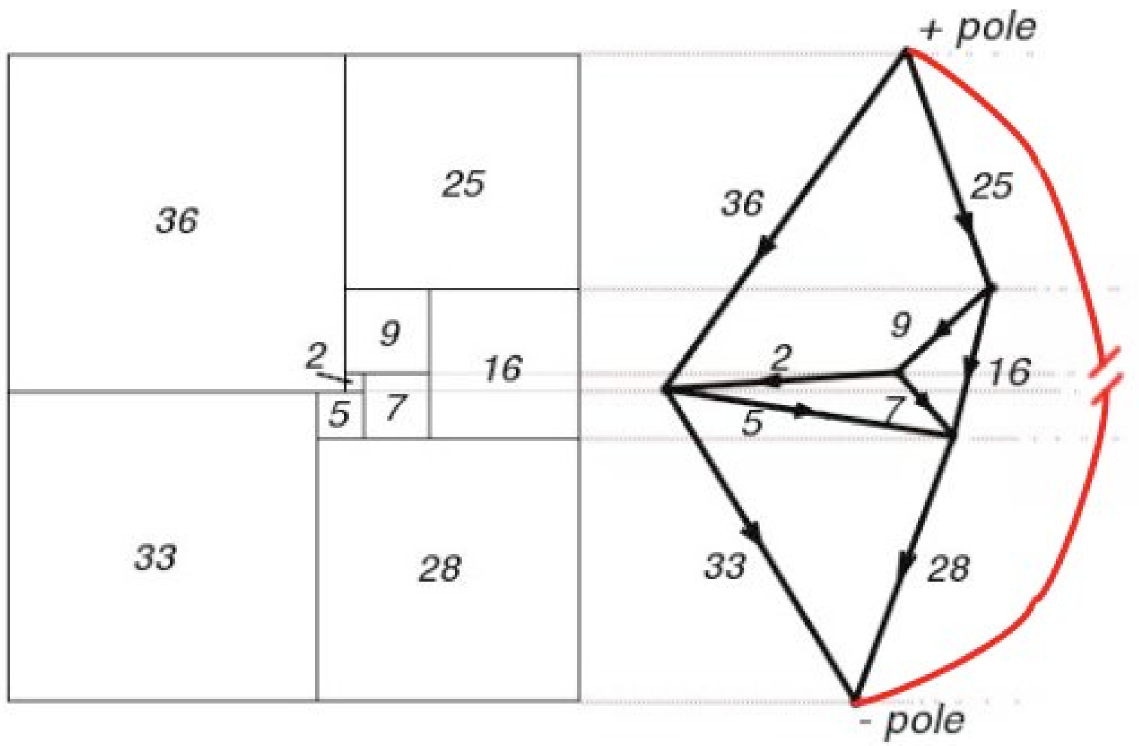
7a = b

If we keep it simple and make a = 1, then b = 7. When we replace these values for the variables in the original equations, we find that the sides of the rectangle measure 32 and 33. This is the perfect squared rectangle found by Moron´ in 1925. In fact, this is the lowest possible order of a perfect squared rectangle, as proved in 1940 by mathematicians named Reichert and Toepkin (Steinhaus).

As an aside, it is of note that the tiled rectangle has a property related to the "Golden Ratio", or phi: the number $\frac{1+\sqrt{5}}{2}$. Moron´ had discovered that any squared rectangle could be enlarged by "adding a square of the same length of side to either side. (squaring.net)." P.J. Federico, a mathematician and patent officer, later realized that enlarging a squared rectangle in this manner, alternating sides, made the connection to phi: Adding a square of the same length of side to either side will make the side length of the new large rectangle equal to the sum of the new square and the one before it. The Fibonacci sequence is obtained by adding the two previous numbers to find the next number. By continuously adding squares and then adding the side length of the new square and the old square to find the side length of the new rectangle, "…the squares correspond to the Fibonacci sequence and therefore the ratio of sides in this infinite sequence approaches phi, the golden mean (squaring.net)."

The Electrical Circuit Solution Method

An important paper on the subject of squared rectangles was published in the Duke Mathematical Journal in 1940. Written by R.L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, the paper summarized the progress of solutions to the problem up to the time the paper was written. The authors cited Max Dehn and others who had made both correct and incorrect conjectures about the problem. W. T. Tutte reminisced in his book, Graph Theory As I Have Known It, on how the four authors, as college students at Trinity, had become interested in the problem. He writes of how A. H. Stone had inferred from Dudeney's statement that his solution was unique that a rectangle could be dissected into smaller squares and rectangles, but a square could not. The four students began studying the problem and using the algebraic solution method discussed above to catalogue various orders of squared rectangles. They were hoping that as they built the catalogue, they were bound to come across a solution that would lead to a squared square (Gardner). They came to realize that the problem was much more difficult than it appeared. Finally, sometime between the years 1936 and 1938, C.A.B. Smith restated the problem of squaring rectangles as that of an electrical circuit. Smith represented each of the horizontal line segments in the drawing of the squared rectangles as a dot. Each dot represents a terminal in the electrical circuit. A line connecting two of the dots (terminals), then, will represent the square that has those two horizontal lines as boundaries. The length of the line connecting the two dots (terminals) is the side length of the square the line represents. This is an example of the Smith diagram, next to its corresponding squared rectangle:



(www.squaring.net/history_theory/gfx/figure73.jpg)

If we picture the top and bottom sides of the rectangle as the poles of an electrical circuit, then we can design the diagram to follow the squared rectangle from top to bottom. Therefore, the square of side length 25 is represented with a line going from the positive, or upper, pole to the next terminal. It is clear that there are nine squares in the large rectangle that correspond to the nine lines that compose the diagram. It may seem difficult to make the connection between the rectangle and the circuit diagram, especially since the smallest square in the rectangle does not correspond to the smallest line in the diagram. However, once one realizes that the squares are each represented by a line, and that the lines make up the rectangle, it becomes quite intuitive to represent the rectangles in this way. In order to make this leap, we must delve into a little bit of electrical circuit theory.

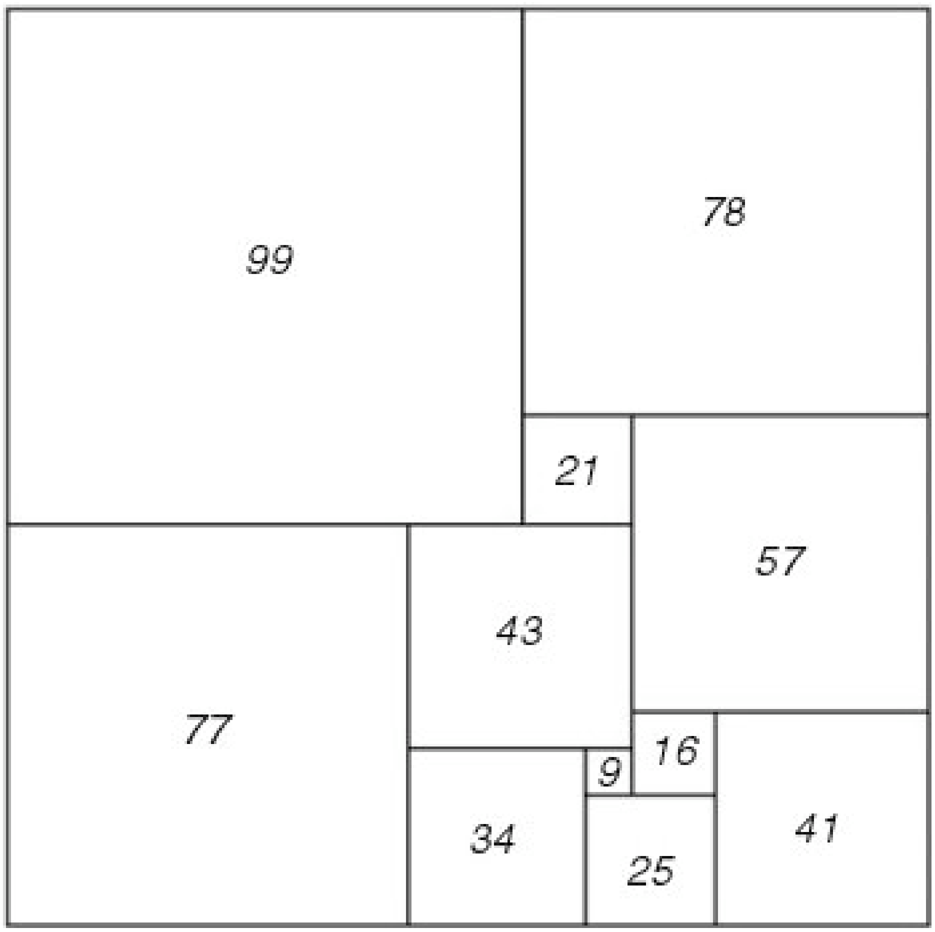
In any electrical circuit, there are laws that describe how the current must behave if the circuit is to be complete. These are known as “Kirchhoff’s Laws.” There are two laws. The first states that the sum of the currents flowing to any of the terminals must be zero. Referring to the diagram above, we can look at the terminal on the outermost left side of the diagram. There are four lines connecting to that terminal. The lengths of those lines are 36, 2, 5, and 33. The arrows represent the flow of the current into and out of the terminal. Incoming current to the terminal is 36 + 2 = 38. Outgoing current from the terminal is 5 + 33 = 38. Therefore the sum of the currents is zero. This is true for every terminal in the diagram, so Kirchhoff's first law holds. Kirchhoff's second law has to do with the entire circuit, and it states that the sum of the currents for the entire circuit has to be zero. In other words, what goes in, must come out. This means that if the positive pole represents the top edge of the rectangle, which is 36 + 25 = 61, then the negative pole represents the bottom edge of the rectangle, which is 33 + 28 = 61. The same amount of current that goes in to the positive pole comes out of the negative pole, so Kirchhoff's second law holds as well.

One might ask how the discovery that squared rectangles could be represented in this way helped the four scholars in their quest for the perfect squared square. In W. T. Tutte's words, "The discovery of this electrical analogy was important to us because it linked our problem with an established theory. We could now borrow from the theory of electrical networks and obtain formulas for the currents in a general Smith diagram and the sizes of the corresponding component squares (Gardner, 192)." In other words, the scholars assigned values to the voltage of the circuit and the resistance, and this allowed them to use Kirchhoff's laws to solve for the ratios of height and width of the tiles, or squares. They realized that "every rectangle tiled by rectangles gives rise to a circuit in which the sizes of both the large rectangle and its subrectangles are determined by the solution to the corresponding circuit laws" and "every connected planar resistive circuit with one battery gives rise to a rectangle tiled by rectangles (Cannon, Floyd, Parry 136)." Using electrical theory, the scholars proved that indeed 9 is the lowest possible order of a perfect squared rectangle, and that there are exactly two such rectangles (Gardner). Their proofs were published in their article in 1940, the same year that Reichert and Toepkin published proof of the same conclusion. One might ask the question, "How does borrowing from electrical theory allow us to determine the number of possible perfect squared rectangles given the dimensions of length and width?" Working with the diagrams of electrical circuits, or polar nets, it can be determined exactly how many perfect square rectangles one can produce with given dimensions. Since there are two distinct polar nets for a diagram with nine wires, there are exactly two perfect squared rectangles of order 9. Since there are six distinct polar nets for a circuit of order 10, there are exactly six perfect squared rectangles of order 10, and so on. Brooks, Smith, Stone, and Tutte began to determine the number of perfect and imperfect squared rectangles of given dimensions. When they reached the 13th order, they realized they had come to the limit of what they could find about squared rectangles using their method by hand (Skinner). Fortunately, before much time had passed, another scholar across the world would use computers to revolutionize the way this problem was finally conquered. A perfect squared square would eventually make its appearance.

The Computer Algorithm Solution Method

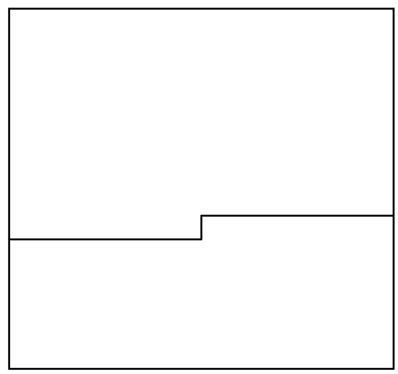
Once one has discovered the dimensions of a squared rectangle using the algebraic or electrical circuitry method, and has drawn the polar net, it is possible to find all combinations of squared rectangles of those dimensions. To do this, each line segment, or wire, on the circuit diagram is removed and the results tested. If there is another location for that segment that will make the polar net valid, then there is another way to arrange the squares within the large rectangle, creating a different solution for a squared rectangle of the same dimensions (Skinner). This method is clearly superior to the algebraic method in terms of speed and accuracy. This method is also one that could be written as computer code, and machines could be made to do the work that was so cumbersome and time-consuming for man - and the machine would be able to do it without flaw.

C.J. Bouwkamp first became interested in the problem of perfect squared rectangles in the early 1940's. His work on diffraction functions eventually led to discovery of the work done by the four students at Trinity College. W.T. Tutte and Bouwkamp actually worked together on the problem for a time, but Bouwkamp was the one who could see the value of using computers to use the electrical circuitry method of solution and accurately catalogue the results. Bouwkamp made up a code to describe squared rectangles, to facilitate the writing of a computer algorithm that could remove one value at a time from the code and test the results, thereby finding all possible solutions to a perfect squared rectangle of given dimensions. Bouwkamp code lists first the order of the squared rectangle, followed by the lengths of the height and width. Then, using parenthesis and sometimes commas, the squares are listed first left to right, and then top to bottom. For example, take the following diagram:



In Bouwkamp code, this squared rectangle would be described thusly:
11 176 177 (99, 78) (21, 57) (77, 43) (16, 41) (34, 9) (25) (http://djm.cc/simple-perfect-rects-to-16.txt). Although the creation of this code was important to the giant leap of using computer code to solve the problem of squared rectangles, a college student working under W.T. Tutte in Waterloo, Ontario, Canada would make the transition go more smoothly. As his dissertation, published in 1967, John C. Wilson took on the problem of squared rectangles. He was looking at simple perfect squared squares to see if they shared a commonality that would allow him to determine whether they could be dissected into two perfect squared rectangles (Skinner). He found that squared squares

that have a “stair step” moving horizontally from edge to edge meet this criterion. The structure looks like this:



and is referred to as a “Wilsonian Structure.” This allowed for computers to search for Wilsonian Structures to obtain simple perfect squared rectangles, AND search for simple perfect squared rectangles to find perfect simple squared squares. At the same time, analysis of the nets allows the computer to catalogue perfect squared squares as a subcategory of perfect squared rectangles.

Conclusion

Mathematicians and puzzlers have been studying this problem for more than 100 years. Has the problem been solved completely? Amazingly enough, there is more to be done! Among the questions being asked is, “Can a plane be tiled with squares such that each natural number is used exactly once as a tile (http://www.daviddarling.info/encyclopedia/S/squaring_the_square.html)?” Such a tile would certainly be magnificent. Mathematicians continue to be interested in the problem and as they work, they find more questions to ask and to answer. Such is the nature of mathematics.

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