Primes in Spirals

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A Thesis Submitted to the Faculty of the

DEPARTMENT OF MATHEMATICS

In Partial Fulfillment of the Requirements
For the Degree of

MASTER OF MIDDLE SCHOOL MATHEMATICS LEADERSHIP

In the Graduate College

THE UNIVERSITY OF ARIZONA
2010
Abstract

Prime numbers have been a fascination to mathematicians for ages. They seem to come in particular form. They do not seem to be distributed in any particular way. There is no one formula that will generate all prime numbers. They do not seem to follow any particular pattern, except maybe when looking at the prime numbers arranged in a spiral.
Identifying Primes

Before looking at primes in spirals, you have to understand prime numbers, what they are, and how to identify them. Prime numbers are positive whole numbers that have only two factors, itself and one. An easy way to identify prime numbers is using the Sieve of Eratosthenes. If using a hundreds chart, begin with two and cross out all of the multiples of two, then continue with three and cross out all multiples of three not already crossed out, and the next consecutive numbers not crossed out will be five then seven. Once you have completed the multiples through seven, you can stop, only primes will be left. Always crossing out the multiples, but not the beginning number. When you have completed this process you are left with only prime numbers, which are highlighted below.

![Primes using Sieve of Eratosthenes](image)

Another way to identify prime numbers is using modulus six. To find primes consider each multiple, $m$, of six and one greater and one less (mod 6+1 and mod 6+5) as possible primes. Test each possibility by dividing it by numbers that are $\leq \sqrt{m}$. All of the red numbers below are primes, with each arm being modulo 6 plus the number inside the circle.
Figure 2. A modulus-6 clock spiral showing the primes (red) to 90
http://www.chass.utoronto.ca/french/as-sa/ASSA-14/article7en.html

**Rectangular Array**

My task was to see what the longest string of prime numbers I could find in a regular rectangular array. I could change the width of the array or what I was counting by. All of the equations for these arrays are linear. There is a constant difference between each prime in the string. When counting by one with the array twenty-nine or thirty-one wide, I was able to find a string of six primes diagonally. There is a difference of thirty between the primes in both arrays. Below is a part of the array that shows the longest string.
Figure 3. Partial 30-column rectangular array string of 6 primes

The equation for this string of primes is $y=30x+7$, this equation is linear because there is a difference of 30 between each prime and you add 7 since that is the first prime in this string. This same pattern also appeared on an array with nine columns, adding three to each number to get the next.

Another array that also has a string of six primes was on a six-column array and adding five to each number. All of the primes are in the first or third column. The beginning numbers for these columns are 1 and 11. Since I was adding five each time all the primes end with one.

The equation is: $y=30x+1$, because there is a difference of thirty between each prime and you add the initial one from the beginning of the column. All of the primes are in the form of mod 6+1 in the first column or mod 6+5 in the third column.
Another array in the form of mod 6+1 or mod 6+5 was a six-column array of consecutive numbers. On this array all primes fell into the column under one or five with the exception of 2 and 3. The equation for that array was $y=6x-1$, (mod 6+5). The longest string had five primes of 5, 11, 17, 23, 29.

On all of the arrays all of the prime numbers within a string, on the given chart work out to be either modulo 6+1 or modulo 6+5, but not both within one array. They do not have the same equations, but the difference between the primes in the string is always a multiple of six. The vertical string of primes on arrays that I found where this was the case also include these: 12 column array adding three to each number with an equation of $y=36x-5$, longest string had four primes of 31, 67, 103, 139; a 6 column array adding three to each number with an equation of $y=18x+7$, longest string had four primes of 43, 61, 79, 97; a 4 column array adding three to each number with an equation of $y=12n-5$, longest string had four primes of 7, 19, 31, 43; and a 6 column array of only odd numbers with an equation of $y=12n-7$, longest string had five primes of 5, 17, 29, 41, 53. The horizontal string of primes on arrays where the mod 6+1 or mod 6+1 applies includes: 12 column array of consecutive numbers with an equation of $y=12n-7$, longest string had five primes of 5, 17, 29, 41, 53 which is mod 6+5; 10 column array of odd numbers with an equation of $y=18x-1$, longest string had four primes of 53, 71, 89, 107 which is mod 6+5; 9 column array of odd numbers with an equation of $y=18+7$, longest string had four primes of 43, 61, 79, 97 which is mod 6+1; 9 column array adding three to each previous number with an equation of $y=30x+7$, longest string had six primes of 7, 37, 67, 97, 127, 157 which is mod 6+1; 11 column array of odd numbers with an equation of $y=24n-11$, longest string had four primes of 79, 103, 127, 151 which is mod 6+1; 6 column array
adding four to each previous number with an equation of $y=24n-11$, longest string had four primes of 349, 373, 397, 421 which is mod 6+1; 7 column array adding five to each previous number with an equation of $y=30x+11$, longest string had four primes of 401, 431, 461, 491 which is mod 6+5; 7 column array of consecutive numbers with an equation of $y=6x+1$, longest string had four primes of 41, 47, 53, 59 which is mod 6+5; and 11 column array of consecutive numbers with an equation of $y=12x-7$, longest string had five primes of 5, 17, 29, 41, 53 which is mod 6+5.

**Spiral Arrays**

Another part of my problem was to find the longest string of prime numbers within a spiral. To spiral the numbers, I started with one in the center then moved out and around creating a spiral with the numbers. Again, I could change my beginning number and what I was counting by. Mathematician Stanislaw Ulam first discovered the spiral array of primes in 1963, while sitting in a meeting. He put the number 1 in the center and began spiraling out from the center with consecutive numbers. He then circled the prime numbers and found they tended to line up in a diagonal. Below is an Ulam spiral that contains a total of 160,000 integers. Prime numbers are the black pixels. There are 14,683 primes in this Ulam spiral.
I began using consecutive numbers with one at the center of the spiral. The longest string was a string of five primes. They did not go through the center of the spiral, nor did they have a consistent first difference, so it was not linear nor did it have a consistent second difference. The second difference was close to being the same, with only five primes in the string it is hard to be positive that the pattern would continue with the second differences being eight. If the pattern continued, then the spiral equation would be in the form of a quadratic. I came up with a quadratic equation that does not fit the first two numbers in the string, but does fit the last three numbers which is

\[ y=4x^2-4x-1. \]
I then continued with one in the center of the spiral building different spirals by adding different amounts to one and each previous number on each spiral. I created eight different spirals with one as the starting point. The longest string I was able to find was when I added six to each previous number within the spiral; I found a string of thirteen prime numbers. These primes did not go through the center of the spiral; they were left of the center. Again, the second difference was not consistent, the second difference of the first three numbers were different than the remaining primes in the string which had a second difference of forty-eight. As you can see in the spiral below the bottom two numbers in the string are not in order. When I dropped the numbers that are out of order, I could work with the remaining numbers to come up with an equation. The equation for this string is, \( y=24x^2+36x+43 \). The primes in green are all modulus 6+1.

Figure 6. Partial consecutive spiral starting with 1.
The spirals that did not go through the center caused me a lot of trouble. I kept trying to get my equation to fit all the numbers in the string, and I could not make it work, because the second differences were not all the same. I asked several people that I know for help and none of them could come up with an equation that would work. I used the table that I had created on my spreadsheet, I highlighted what my x and y axis would be and inserted a chart. Once the chart wizard opened, I selected a scatter plot. When the chart was created, I right-clicked on one of the data points and selected add trendline, I used a polynomial trend/regression line and showed the equation. When I finally dropped the beginning primes and focused on the ones that had a consistent second difference, I was able to generate an equation that would work for the remaining primes in the string.

I kept creating different spirals with one in the center and never came up with any with a longer string of primes than thirteen. I finally decided to try different numbers other than one in the center and see what happened. I started off with placing eleven in the center and using consecutive numbers, this created a string of only ten primes in a string. All of these primes had a second difference of two, and the string went through

Figure 7. Partial spiral array starting with 1 and adding 6
the center of the spiral. I then tried starting with seventeen, I found a string of sixteen primes, and again all the primes had a second difference of two, and went through the center of the spiral. Next I created a spiral starting with thirty-five. This spiral had a string of twenty primes; the string did not go through the center of the spiral. This was another spiral that caused me a lot of trouble. The first six primes in the string do not fall in order because of the spiraling. After the prime 173, the primes all fall in order and have a difference of 8. Again, once I dropped the first six prime numbers in this string, I was able to come up with an equation of $y=4x^2-2x+41$ that would give me the rest of the prime numbers in the string.

Another of the spirals I tried started with forty-one in the center, which has a string of forty primes through the center. This was the longest string that I found. The second difference of this string is two and the equation is $y=x^2-x+41$. 

Figure 8. Partial spiral starting with 35.
**Mathematical Ideas**

It appears that when the string of primes goes through the center, the second difference remains constant throughout, when it does not go through the center the second differences are not constant in the beginning. The four spirals where I started with numbers other than one in the center all have the same pattern when looking at modulus 6 which is: \(\text{mod } 6+5, \text{mod } 6+1, \text{mod } 6+5, \text{mod } 6+5, \text{mod } 6+1\ldots\)

After finding a modulus pattern in the rectangular arrays first, I then looked for modulus patterns in the spirals, which I found. Although there are patterns within modulo and primes, there are no patterns within primes themselves. There is no one equation that will generate prime numbers. Within all of the different arrays I created, there are an equal number of equations, with each equation specific to a certain array. There are similar forms of equations, but no one specific equation will work for all.
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Figure 2: http://www.chass.utoronto.ca/french/as-sa/ASSA-14/article7en.html

Figure 5: http://mathworld.wolfram.com/PrimeSpiral.html
Appendix

primes.xls