Cultivating Symbol Sense in Your Calculus Class

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1 Introduction

Express

$$\lim_{||P||\to 0} \sum_{i=1}^{n} \left[(x_i^*)^3 + (x_i^*) \sin (x_i^*) \right] \Delta x_i$$

as an integral on the interval $[0, \pi]$. [1, p. 338]

Many students might find this task a bit overwhelming. Why? Because it is highly symbolic, it might contain notations that are unfamiliar to them, and it might employ unfamiliar conventions. But even more, the symbolic formulation in the task is dense with meaning, and students are often disinclined to unpack a statement's meaning. Even when they try to unpack meaning, they often find it difficult. When students do not understand what a symbolic expression means, they often resort to meaningless "symbol pushing," which is both typical and, in the long run, counterproductive. This example is revisited later.

In order to succeed in calculus, students must make connections among concepts and between concepts and symbolic expressions of them. Unfortunately, many students enter calculus with a long history of experiencing mathematics as "rules without reason on marks without meaning."

The above statements are grounded in research:

- People use symbols to convey meaning compactly. Students' development of symbolic facility is therefore highly intertwined with their development of the meanings they attempt to convey symbolically [2, 3].
- To use symbols thoughtfully, students must practice expressing their thinking coherently, then use symbols to generalize their thinking and express it more compactly [4]. When students fail to develop coherent meanings, symbolic expressions of them become "marks without meaning" and, to cope with demands that they perform, students develop "rules without reason."
- U.S. teachers often short-circuit the above process by introducing symbols too early and do not consistently use them meaningfully [4]. This has been known for a long time [5].
- The meaninglessness of students' mathematics increases steadily with the number of years they have been in school [6, 7].

2 Why students have trouble with symbols

When students have a problem with symbols, it is almost always due to a problem with the students' meanings. Students might be able to manipulate expressions and equations, but they often don't attach meanings to those symbols, or attach vague, imprecise meanings. When this happens, they will likely fail to conceptualize calculus properly and appreciate its importance. Too often, a symbolic expression isn't saying something meaningful to the student misusing it. Without an understanding of the fundamental concepts of calculus, students are easily overwhelmed in subsequent college math courses that are taught more theoretically.

Part of the issue lies in the fact that many students develop the view that symbols are cues for algorithms rather than representing a mathematical idea. For example, in a basic algebra unit on exponents, many students learn that a superscript of "-1" on an expression means "1 over" that expression. They often cannot break from that ingrained meaning of the superscript -1. When used with a function, such as a trigonometric function, "-1" as a superscript takes on a new meaning; namely, that of inverse function. Upon seeing $\sin^{-1}(x)$, students might immediately, but incorrectly, think of $\frac{1}{\sin(x)}$ or, even worse, $\frac{1}{\sin}(x)$, instead of $\arcsin(x)$. To avoid such confusions, students should be asked repeatedly to compare and contrast the two meanings of the superscript "-1" so that they do not confuse them when in the midst of interpreting a mathematical statement that uses it.

Many symbols mean different things in different contexts. Uses of parentheses are a frequent source of confusion. In pre-algebra, students learn that parentheses indicate the product of two expressions. When function notation is introduced, they often interpret $f(x)$ as the product of a number f and a number x. The meaning of $f(x)$ as a rule (represented by f) applied to a number (represented by x) is unlike anything they have met before. Representing a rule by a letter is, to students, very strange when in their experience letters have always represented numbers. In a similar vein, students learn in algebra that an ordered pair, like $(2, 3)$, represents a point in the coordinate plane specified by its coordinates. That becomes the meaning that jumps to mind when they see an ordered pair. In calculus, what looks like an ordered pair now can represent either an interval of numbers or a point in the plane. To avoid confusions, students must learn to be sensitive to the context in which the notation "$(2, 3)$" appears. A third example is uses of an apostrophe. An apostrophe after a symbol, such as f', might be intended to indicate the image of a line named f that has undergone a transformation, or it might denote the derivative of a function named f. When students fail to give meaning to a symbol by drawing upon the context in which it occurs, they often give up on developing understanding of the symbols. Instead, they simply look for clues as to what algorithm the symbol suggests.

While context is often an issue, the symbols themselves may confuse students. For example, the rate of change of a function is often symbolized by $\frac{dy}{dx}$. To students who do not interpret dy and dx as differentials that are tied by the relationship $dy = f'(x)dx$, this might look suspiciously like a rational expression in which the d's can be cancelled. Additional confusions can arise even when students understand dy and dx as differentials – especially when the text (or instructor) insists that $\frac{dy}{dx}$ is not a ratio but is instead a symbol that represents a function's derivative. Throughout these examples, the main issue is that students will often resort to "rules without reason on marks without meaning" without the instructor's clear attention to symbols' meanings in context.

3 Vignette

In the opening, we posed the problem

Express

$$\lim_{||P|| \to 0} \sum_{i=1}^{n} \left[(x_i^*)^3 + (x_i^*) \sin(x_i^*) \right] \Delta x_i$$

as an integral on the interval $[0, \pi]$.

Let us imagine how an AP teacher might help a class of students understand all that this task entails. We should note that this AP teacher has established the practice of discussing symbolic expressions' meanings and regularly insists that students attempt to interpret such expressions as they read their text.

T: On page 338, Stewart poses this problem. It contains this really complicated symbolic expression, and I think it will be worthwhile if we pause to understand all that this expression says. Now, you're probably tired of me reminding you, but remember that any statement contains parts, those parts have meanings, and a statement's overall meaning comes from piecing together the meanings of all its parts. So let's talk about the parts of this statement and their meanings.

S1: I don't understand the P with bars around it.

S2: The P stands for "partition." It's when you cut up an interval into a bunch of pieces. P stands for all of the pieces.

S3: And the bars say how to think about the size of the pieces. It's like "P-with-bars" equals the length of the biggest piece in the partition.

T: Good. We're talking about a limit as we take successive partitions where the biggest piece gets smaller and smaller. Now, someone tell me what happens to the number of pieces that we cut an interval into if the size of the biggest piece goes to 0?

S4: In order for the biggest piece to get really small, you have to cut up the interval into a lot more pieces.

T: Good, S4. And where in this statement does Stewart mention the number of pieces?

S1: Oh! With the "n" at the top of the summation.

T: Good, S1. Now, we've focused on what the "P-with-bars" means and how that fits in with n, but what about the overall statement? S5?

S5: Oh, I see. We're taking the limit of something as the number of pieces gets larger.

T: Can you be more precise about what you mean by "something"? [Long pause.]

S2: The something is a sum of areas.

T: And what are those areas? How are they defined? S6?

S6: The bits of area are made by rectangles. A rectangle's width comes from the interval it sits on, and the rectangle's height comes from the function.

T: Okay, you've raised two good points. Indeed, we are summing the areas of rectangles. But how do we *represent* the width of each rectangle, and how do we *represent* each rectangle's height?

S7: Oh, the rectangle's height is $(x_i^*)^3 + (x_i^*)\sin(x_i^*)$. And the rectangle's width is Δx_i^*.

S8: What does the "x star" stand for?

> *The discussion continues. In the interchange it is made clear that (a) the function being integrated is $f(x) = x^3 + x \sin x$, and (b) that Stewart's approach is to make each rectangle have a height that is the value of f at some arbitrary value within the respective subinterval.*
>
> *The teacher ends the conversation by distributing a handout with the expression $\lim_{||P||\to 0} \sum_{i=1}^{n} \left[(x_i^*)^3 + (x_i^*)\sin(x_i^*)\right] \Delta x_i$ on it, and the directions "circle the major parts of this statement, say what they mean, and say what the entire expression means. Then, express $\lim_{||P||\to 0} \sum_{i=1}^{n} \left[(x_i^*)^3 + (x_i^*)\sin(x_i^*)\right] \Delta x_i$ as an integral on the interval $[0, \pi]$."*

The teacher in this vignette made several important decisions. She drew students' attention to strategies for reading symbolic expressions, she focused students' attention on the expression's syntactic structure, she insisted on precision of meaning and of expression, and she insisted that many students contribute to the discussion. Such practices, over time, have a positive effect on classroom culture. Over time, a classroom norm emerges – students come to expect to make meaning, to express meaning symbolically, and to read meaning from symbolic expressions.

4 Role of the teacher

Everyone carries his or her own personal definitions of mathematical concepts – what Tall and Vinner [8] call one's "concept image." Students' concept images often align poorly with the concept's standard mathematical definition, and many difficulties arise when this is the case. Over time, hopefully, their concept images will become consistent with the concept's standard definition – but this does not happen without students reflecting on symbols' meanings in use, and it is the instructor who establishes the mathematical habit of reflecting on meanings.

Calculus instructors can assist students in using symbols meaningfully by changing the culture of their classrooms. Here are some ways instructors can do that.

1. **Meanings before symbols**

   Because an instructor already has symbolic understanding, she might be inclined to introduce symbolism as she teaches a new concept; however, if students are encouraged to develop meaning first – before it is symbolized – then, when the symbol is introduced, students will view it simply as an abbreviation of the concept rather than as an indicator of an algorithm. Instructors can motivate students' appreciation for symbolism by highlighting its usefulness. After a symbol or symbolic expression is introduced, a teacher who has her own classroom might exhibit it, along with its meaning, on a bulletin board or a "word wall" display.

2. **Speak meaningfully**

   As an exercise, a mathematics teacher might occasionally have students translate back and forth between a mathematical expression entirely in words and one that is written symbolically. For example, students might practice stating a meaning for average rate of change over an interval that entails the net change in the function's value in relation to the size of the interval. In symbolizing this meaning, they should represent the interval, such as $(x, x+h)$, the interval's size as h, and the net change in the function's value over that interval as $f(x+h) - f(x)$.

3. **Address difficulties with symbols explicitly**

   One way to help students with potentially confusing symbolism is to offer historical insight into the development of those symbols. For example, a story about the development of Leibniz notation might help students understand the dx in integral notation.

   Another way to alleviate confusion is to explicitly point out to students that symbols often have different meanings in different contexts, and that alternate symbolism often exists with the same meaning. For example, the superscript -1 discussed earlier has a different meaning (from a beginning student's perspective) when used with a variable than it does when it is used with a function. Another example is the many symbols we use to represent a function whose values are the instantaneous rates of change of another function: $\frac{dy}{dx}$, $\frac{df}{dx}$ where $y = f(x)$, $f'(x)$, and $\dot{x}$.

4. **Unpacking complex symbolism**

   The teacher in the vignette "unpacked" the expression piece by piece. Unpacking the meaning of a symbolic expression includes parsing the expression into smaller reference units. Not only would a teacher of calculus model this unpacking process; he would also encourage students to do the same. Care should be taken so that students comprehend a reference unit. For example, in the expression $(x_i^*)^3 + (x_i^*)\sin(x_i^*)$, "$\sin(x_i^*)$" is a reference unit, whereas "sin" alone is like a phoneme rather than a full word. By habitually unpacking symbolic statements' meanings, students can more readily attach meaning to symbols and extract meaning from symbolic expressions.

Calculus teachers will find that a new culture emerges in their classrooms when they are conscientiously and consistently sensitive to students' meaningful use of symbols. Students will make connections between concepts of calculus and the symbolism used to represent these concepts. As a byproduct, students will develop symbol sense and will become better symbolic reasoners.

References

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