Problem Solving

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1 Introduction

Calculus students spend a lot of time on exercises and problems. Exercises may be useful or essential in their own right, but this pamphlet is about problems. The key difference is that problems are deeper and more challenging than exercises, and normally involve some element of novelty.

Here are some examples to illustrate the broad range of possibilities. We have chosen relatively "pure" examples; applied problems raise additional issues.

Problem 1: *Let f and g be the functions shown graphically below, and let $h(x) = (f \circ g)(x)$.*

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- **(a)** *Evaluate $h(-2)$, $h(1)$, and $h(3)$.*
- **(b)** *Estimate $h'(-2)$, $h'(1)$, and $h'(3)$.*
- **(c)** *Is h increasing at $x = -1$? How do you know?*
- **(d)** *Which values of x correspond to critical points of $h(x)$?*

To decide whether the composite function $h(x) = (f \circ g)(x)$ is increasing at $x = -1$, we can consider the derivative $h'(x)$, and try to decide whether $h'(-1)$ is positive or negative. Because h is a composite function, we expect to use the Chain Rule: $h'(x) = f'(g(x)) \cdot g'(x)$. In particular, $h'(-1) = f'(g(-1)) \cdot g'(-1)$.

Now we look at the graph of g. The slope of the tangent line to the graph of g at $x = -1$ is negative, and so $g'(-1)$ is negative. The graph of g shows, too, that $g(-1) \approx 3$, so we need to find $f'(g(-1)) \approx f'(3)$. The graph of f shows that $f'(3)$ is also negative. Thus $h'(-1)$ is the product of two negative numbers, and is therefore positive. Thus h is increasing at $x = -1$.

Another possible approach involves looking at the graph of g and noticing that if $x = -1$, a small increase in x produces a small decrease in $g(x)$. Looking at the graph of f, we see that if $g(x)$ is about 3, as it is when $x = -1$, a small decrease in $g(x)$ produces a small increase in $f(g(x))$. Therefore, a small increase in x ultimately produces a small increase in $(f \circ g)(x)$. This covariational perspective can be used not only to produce an alternative solution to this problem, but also to reinforce the conceptual underpinnings of the Chain Rule.

Problem 2: *Does the function $f(x) = x^{21} + x^{19} - \frac{1}{x} + 2$ have any roots between $x = -1$ and $x = 0$? Justify your answer.*

This problem challenges students' first assumptions and their tendency to try only one approach. A student might, for example, expect to be able to use the Intermediate Value Theorem. She could evaluate $f(-1) = 1$, but would find that $f(0)$ is undefined. She might next evaluate, say, $f(-0.1)$, but would then obtain another positive value. Thus the Intermediate Value Theorem is not immediately applicable.

One student might next turn to a graphing calculator, to see that $f(-1) = 1$ and that the graph appears to rise on the interval $(-1,0)$. To make this rigorous, the student could calculate $f'(x) = 21x^{20} + 19x^{18} + \frac{1}{x^2}$. The formula looks complicated at first sight, but because all powers of x are even, each summand is positive, and so $f'(x) > 0$ for all x in $(-1,0)$. Now the first derivative test implies that – as the graph suggests – f is increasing on $(-1,0)$. Since $f(-1) = 1$, the function f has no roots in $(-1,0)$.

Alternatively, a student may make the observation that each of the functions x^{21}, x^{19}, and $-\frac{1}{x}$ is increasing on the interval $(-1,0)$ (indeed, the first two are increasing on the entire real line). Therefore, the sum of these three functions with a constant is itself an increasing function on $(-1,0)$. Since $f(-1) = 1$, the function cannot have any roots in the interval $(-1,0)$.

Problem 3: *Find the minimum value of* $f(x) = x^2 + 2x\sin x - \cos^2 x + 1$.

Many calculus students would, naturally enough, approach this problem using calculus techniques.

Here is one such approach. Applying the power rule, product rule, and trigonometric rules for derivatives gives a formula for $f'(x)$, which can be factored:

$$f'(x) = 2x + 2\sin x + 2x\cos x + 2\cos x\sin x = 2(x + \sin x)(1 + \cos x)$$

Solving $f'(x)$ gives the critical points $x = 0$ and $x = k\pi$ for odd integers k. Further work with the first derivative test, or checking values explicitly, reveals that $f(0) = 0$ is the minimum value in question.

Calculus methods, however, are not the only possibility. As some students may notice, a trigonometric substitution reveals a perfect square:

$$f(x) = x^2 + 2x\sin x - \cos^2 x + 1 = x^2 + 2x\sin x + \sin^2 x = (x + \sin x)^2.$$

This form reveals that $f(x) \geq 0$ for all x. Since $f(0) = 0$, the minimum value is attained at $x = 0$.

2 Themes

Several themes emerge repeatedly as we think about problem solving and the difficulties it raises in calculus courses.

2.1 Mathematical language and precise communication

Calculus is a subtle subject. Communicating calculus ideas requires more careful use of language than many students are accustomed to. Here are some examples:

1. Solving challenging problems requires students to understand, manipulate, and "translate" complicated mathematical and linguistic constructions. They need, for instance, to distinguish clearly between "implies" and "is implied by."

2. Many problems ask students to explain or justify something. Doing so requires both mathematical and linguistic facility, and awareness of one's own reasoning processes.

3. Definitions in calculus are intrinsically subtle; understanding them requires careful parsing of technical language.

4. Challenging problems are seldom solved in a single step. Students must learn to assess both what is being asked and what is known.

2.2 Drawing connections

Solving interesting problems may require students to combine several ideas and methods – not necessarily all from one section or chapter in a book. Doing so may be challenging:

1. Students often rely too narrowly on symbolically-presented functions and symbolic operations on them.
2. Many calculus ideas, such as instantaneous rates of change, are closely analogous to physical phenomena, such as speed. Students may have useful physical intuition, but find it difficult to relate to mathematical ideas.
3. Calculus problems may be presented in symbolic, graphical, tabular, or verbal form. Students find it difficult to connect and move among these various representations.

2.3 Mathematical content

Solving challenging problems may require broad and deep mathematical and meta-mathematical knowledge. Here are some areas in which many students struggle:

1. Students' facility with algebra may be so poor that apparently routine algebraic steps in a complex problem doom a solution and distract students from the main calculus ideas.
2. Many students misunderstand logical underpinnings of calculus ideas, such as the difference between necessary and sufficient conditions.
3. Students may not expect mathematical content from one section of a course to be useful or required in another.

2.4 The affective domain

This domain includes values, beliefs, attitudes, emotions, and feeling. It can play a considerable role in non-routine problem solving.

1. It would be helpful if students believed that exploration, guessing and checking, and thought experiments were both permissible and helpful.
2. It would be useful if students believed that problem solving includes reviewing or reflecting on answers, which should make sense in the problem's context, and that a problem may have more than one answer.
3. Students should ideally learn to employ good problem-solving strategies as a matter of habit, not conscious thought.
4. It would be good if students could develop and educate a sense of being on the right or wrong track to a solution. They should develop this habit through acting on such feelings and trying different directions.
5. Students may hold various beliefs directly detrimental to problem solving. Here are several:
   (a) No problem should take more than five minutes to solve.
   (b) Non-routine problem solving requires a special knack.
   (c) If problems are difficult, teachers are at fault.
   (d) Hard mathematical problems are mainly long calculations.
   (e) Quick reading should suffice to understand a problem.

3 Role of the teacher

We cannot expect students to write perfect solutions on the first try. In fact, a great deal of scaffolding on the teacher's part may be necessary. Modeling formal solutions during lecture and discussion is a good way to help students understand and meet the teacher's expectations. Furthermore, students should be encouraged to read and assess each other's work for mathematical correctness and for understandable exposition of the solution. The teacher can provide guidelines for this assessment. For example, the student could be charged with reading a classmate's work, checking for pronouns with unclear antecedents, or for a complete justification of a solution.

1. Teachers should provide scaffolding, and give students a temporary structure that they can eventually modify and internalize.

2. Teachers should model good thought processes and provide time and support for student inquiry.

3. Teachers should frame the learning process and help students summarize.

4. Teachers should pose questions and should have students answer them.

5. Teachers should facilitate students' questioning of each other and of their own work.

6. Teachers should acknowledge different approaches.

Suggestions for teachers

Following are some practical suggestions for helping students solve problems more effectively.

1. Require formal write-ups of some solutions. (The problems at the beginning of this pamphlet provide some examples.)

2. Have students peer-review each other's solutions for clarity, reasonableness, precise use of mathematical language, and notation.

3. Assign novel problems that challenge students to apply knowledge in non-routine or unexpected ways. Problem 1, for example, will challenge even students who are familiar with the Chain Rule and the relationship between a function and its derivative. The functions in question are given graphically rather than by formulas, and so the student must consider carefully how to identify input values and output values, how to estimate values of the derivatives, and how to call to mind all of the pieces necessary for a complete solution.

4. Have students write their own problems to help clarify concepts.

5. Create worksheets that contain problems from several sections, to help decompartmentalize student understanding.

6. Provide a structure for brainstorming useful ingredients and possible strategies before beginning to solve a problem. In Problem 1 above, for example, students may know what is being asked, but not how to proceed in the absence of any formula. Some students may also resist having to estimate derivative values from a graph.

7. To help students get started, invite students to compile everything they know that might be relevant to a problem. In Problem 1 above, students might list what they know about composition, evaluating functions given graphically, the Chain Rule, the first derivative test, and graphical estimation of derivatives. The teacher can then ask leading questions to help students apply what they know. For example: "How can we tell whether h is increasing at $x = -1$? How does this relate to the derivative h'? Do familiar rules apply? How?"

8. Require students to state solutions precisely, avoiding pronouns with unclear antecedents. For example, students might critique and peer-edit this sentence: “It is positive at $x = -1$; therefore it increases at $x = -1$.” Encouraging students to write more words – more nouns and fewer pronouns, in particular – can both teach students to communicate mathematically and allows teachers to pinpoint and address misconceptions and errors.

9. Assign multi-day problems to emphasize that problem solving takes time and exploration.