# Limits

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## 1 Introduction

The concept of limit is central to the study of calculus. Limits underlie the two most important concepts in calculus: the derivative and the integral. If a student does not have a robust understanding of limits, this can pose a threat to his future ability to understand the mathematics behind how derivatives and integrals are computed.

The teaching of limits entails certain issues which teachers must be prepared to address. For example:

1. Students often exhibit procedural fluency with limits even though they do not understand what limits mean. For example, when asked to compute the limit $\lim_{x\to 2}(6x^2 - x)$, a student may simply plug in $x = 2$ and arrive at the correct answer without understanding why it is possible to do so.

2. Students who have prior experience with calculus may have techniques for computing limits in certain situations, but may not understand the reasons such techniques work. For example, a student given the limit
$$\lim_{x\to\infty} \frac{4x^2 + 3x + 2}{3x^2 + 1}$$
may know that the limit in this case is the ratio of the leading coefficients of the numerator and denominator, or $\frac{4}{3}$ in this case. However, th student may not understand any of the mathematics behind this shortcut, and may attempt to apply it in a situation in which it does not work, such as the limit
$$\lim_{x\to\infty} \frac{4x^2 + 3x + 2}{3x + 1}.$$

3. Students may occasionally obtain a correct limit for incorrect reasons. For example, given the limit $\lim_{x\to 0} \frac{\sin x}{x}$, students may reason that substituting $x = 0$ yields $\frac{0}{0}$, and therefore the limit is 1 because the numerator and denominator are equal.

## 2 Using derivatives to motivate limits

One motivation for the introduction of limits can be the study of the velocity of an object. One can show that the instantaneous rate of change of a function at a point depends on the behavior of that function in close proximity to that point, and therefore we can use the behavior of the function on smaller and smaller intervals around that point to obtain better and better estimates for the instantaneous rate of change. This kind of estimation is at the core of the concept of limit.

This approach comes with its own challenges. Students may realize that instantaneous velocity, for example, can be read from a speedometer at any given moment, but they may not realize that this number is related to average rates of change. They may not realize that instantaneous velocity is defined not only by the position of an object at a given time, but also the position of the object at other times before and after the time at which they are measuring the velocity. Some students may not think of velocity as a rate at all; they may not see the two basic quantities, distance and time, that define speed, but only the speed itself.

# 3 Eventual limits

We realize that different textbooks use different approaches. While some prefer to first introduce limits by estimating the instantaneous rate of change via a limiting process applied to the difference quotient, which we prefer, other books introduce the notion of limits before the derivative.

Due to differences in how textbooks present material, motivating limits through the derivative may not be practical if one follows the textbook. A possible alternative is to use "eventual limits," by which we mean limits at infinity or limits at points where a function is discontinuous, to motivate the idea of a limit at a point. This perspective is also useful with textbooks that introduce the notion of derivative before the notion of limit, as a way of enriching students' experiences with less routine examples.

There are three types of eventual limits:

1. A limit as $x$ approaches infinity of a function. For example, students might calculate

$$\lim_{x\to\infty} f(x) \qquad \text{where} \qquad f(x) = \frac{2e^{2x}+5e^{-x}}{3e^{2x}-e^{-x}},$$

   and use this information to find a horizontal asymptote of the graph of $f$.

2. A limit as $x$ approaches $a$ of a function which approaches positive or negative infinity. For example, students might compute

$$\lim_{x\to 3^+} f(x) \qquad \text{where} \qquad f(x) = \frac{x(x-5)}{x-3},$$

   and use this information to determine the behavior of $f$ near a vertical asymptote.

3. A limit as $x$ approaches $a$ of a function which has a limit at $a$ but is not defined at $a$. For example, students might compute

$$\lim_{x\to 0} f(x) \qquad \text{where} \qquad f(x) = \frac{\sin x}{x},$$

   and use this information to show that $f$ has a removable discontinuity at 0.

We now discuss these types of limits in detail:

## 3.1 Limits at infinity

We start with this type of limit because it is more natural and accessible to students because of their experience finding horizontal asymptotes of functions, and because the idea of long-term behavior has intuitive value to people.

There are several interesting types of long-term behavior, including $\infty - \infty$, $0 \times \infty$, and $1^\infty$. We illustrate here an example of the latter type. The familiar $Pe^{rt}$ formula for interest compounded continuously arises via passage to a limit from the formula $P(1+\frac{r}{n})^{nt}$ for interest compounded at discrete intervals. In order to understand this, students must first be aware that the limit

$$\lim_{n\to\infty}(1+\frac{r}{n})^n$$

is a limit of the form $1^\infty$, but that the limit of the expression is not 1, as many students incorrectly guess.

This issue can be explored in several ways: by reviewing previously learned facts, graphically, and through a table. Even though an algebraic approach based on L'Hopital's Rule may be possible at this point, using this technique now will likely mask the conceptual understanding this example is trying to convey. Hence algebraic approaches based on L'Hopital's Rule should be deferred until later.

A useful approach based on a table might include three columns: one for the value of the base, one for the value of the exponent, and one for the value of the expression. The table will show the predicted behavior for the base and exponent, but unpredicted behavior for the value of the resulting expression. Follow this with graphing to solidify the observations made in the table.

## 3.2 Limits at a point

The goal of instruction should be that students come to understand that the reason one looks at the limit of a function at a point is because there is something interesting about the function at that point. A limit is interesting when substituting the limiting value in the function results in a situation such as $\frac{a}{0}$, $\frac{0}{0}$, or $0^0$. We will not be exhaustive about these cases but rather illustrative about what an instructor could do to foster the creation of a robust concept image of limit by students while solving specific problems. We recognize that creating that image entails potentially addressing and resolving misconceptions about limits that students might have, and that these misconceptions may not always be obvious.

A good example that illustrates how students may get correct answers by reasoning incorrectly is the case of the limit $\lim_{x\to 0} \frac{\sin x}{x}$. Here are three possible (and often common) student answers:

**Student A:** "The limit is 1."
**Student B:** "The limit is 1, because $\frac{0}{0} = 1$."
**Student C:** "The limit is undefined because you can't divide by zero."

Based on these answers, it is likely that Student C is getting the wrong answer ("undefined") and is also reasoning incorrectly. Student B is getting the right answer, but also arriving at this answer through incorrect reasoning. Student A, on the other hand, may have arrived at the correct answer through correct or incorrect reasoning, and further probing is required to find out which is the case.

To address the faulty reasoning of Student B, we might use a second related example such as $\lim_{x\to 0} \frac{\sin(2x)}{x}$. A graph of the function $\frac{\sin(2x)}{x}$ near $x = 0$ can be used to show that the limit in this case is not 1, despite the fact that we still arrive at $\frac{0}{0}$ if we reason as Student B did in the first example. The graph will suggest that the correct answer in this case is 2. At this point, the student may intuit that the limit $\lim_{x\to 0} \frac{\sin(nx)}{x}$ is $n$. Moving to different examples is now necessary to highlight that limits are unpredictable, and that more exploration is needed to further and complete one's notion of the concept of limit. Some good follow-up examples could be $\lim_{x\to 0} \frac{10^x-1}{x}$ and $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Note that tables can also be used to illustrate what might be interesting about the behavior of a function near the point under consideration. The behavior of the graph of a function near a point should be consistent with the behavior that the table shows.