The Fundamental Theorem of Calculus
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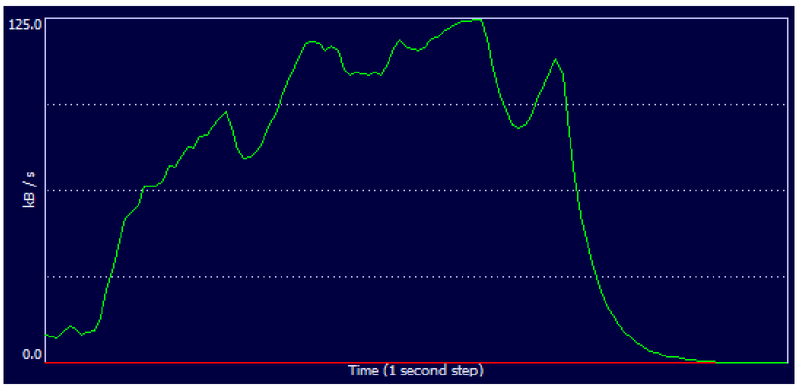
1 Introduction

The following interesting example prepares the way for an intuitive understanding of the Fundamental Theorem of Calculus (FTC). The following graph from `http://samjshah.com/2008/07/29/integration-as-accumulation/` shows download speed as a function of time for a particular file. Each horizontal dot represents one second. How big is the file?



Examples like this should help students develop the understanding that

The definite integral of the rate of change of a quantity over an interval of time is the total amount of that quantity accumulated in that interval.

On the one hand, if $f(t)$ is the amount of a quantity at time t, so that $f'(t)$ is its rate of change, then $f'(t)dt$ can be interpreted as rate $\times$ time, giving the total accumulation of the quantity over a small time interval dt. Using units of measurement makes this vivid; for example, if $f(t)$ is the number of feet an object has traveled after t seconds, then $f'(t)$ is measured in feet/second, and $f'(t)dt$ is measured in feet/second $\times$ seconds = feet. Accumulating these small distances traveled from some initial time $t = a$ to some final time $t = b$ gives the total displacement. The result of the accumulation process is the definite integral

$$\int_a^b f'(t)dt = \text{The total amount accumulated between } t = a \text{ and } t = b.$$

On the other hand, the total amount accumulated in that interval is the difference between the starting and ending quantities, $f(b) - f(a)$.

Comparing these two ways of finding the total amount accumulated, we get the Fundamental Theorem of Calculus:

$$\int_a^b f'(t)dt = f(b) - f(a).$$

2 Key issues

We consider the Fundamental Theorem of Calculus in the form

$$\int_a^b f'(t)dt = f(b) - f(a), \text{ where } f \text{ is a differentiable function.}$$

One important thing for students to understand is that this is a theorem and not a definition. The two sides of the equality have *a priori* quite different meanings, and it is a miraculous fact that they are equal. This miracle allows for the computation of the left hand side. Often the miracle is lost on students because the left hand side does not have meaning for them independently of the computation. The Fundamental Theorem is reduced to a computation without any meaning attached to the thing being computed.

3 Research

The literature dealing with student understanding of the Fundamental Theorem of Calculus suggests that notation and conceptual understanding are both significant. Notation is often interpreted as an afterthought [2]. Notation is also not seen as being related to concepts [1]. Students with a working image of function as an equation of the form $f(x) = \ldots$ which is represented with a graph may have difficulty gaining conceptual understanding due to a lack of understanding of covariation of variables. Covariation is the idea of considering how one variable's change affects the change of another variable. Not thinking about covariation is related to a conceptual cause of trouble – an impoverished understanding of rate of change [2, 1]. As a result, students often do not have a scheme for average rate of change.

4 Teaching suggestions

In order to help students develop a conceptual understanding of the Fundamental Theorem of Calculus which will support both computation and problem-solving, we recommend that calculus teachers incorporate the following suggestions into lessons on the Fundamental Theorem of Calculus.

1. **Use a good intuition-building example.**

   The introductory example above shows the power of a good – and interesting – example to illuminate for students the main facets of the Fundamental Theorem. Take time in the development of this example, and then give students others of their own to develop.

2. **Make a clear decision about whether to prove the theorem or not.**

   For students at this level, a conceptual approach rather than a full epsilon-delta proof is likely to be more productive. However, whether an instructor decides to prove it or not, it is critical that concrete examples like the one above be used to make the concept compelling and clear. A generalization of such a procedure may be sufficient.

   It is also extremely important that students realize that the Fundamental Theorem of Calculus is a *theorem* and not a definition of a method for evaluation of definite integrals. Whether they see the proof or not, it is important that they know that one exists and that it is critically important to the development of calculus.

3. **Let Riemann sums and their limits help to eliminate several student misconceptions.**

   The following statement may help students clarify what the Fundamental Theorem of Calculus accomplishes:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \underbrace{\int_a^b f(x)dx = F(b) - F(a).}_{\text{Result of FTC}}$$

   Riemann sums give students a way to visualize a net change as a sum of smaller changes, as is seen in the introductory example given above. But they also make sense of many contexts that students will see as they develop concepts and skills in applications associated with integration. For example, in working with volumes of revolution, using differential notation such as $dV = \pi r^2 dx$ reminds students that an integral is a limited sum of quantities created as products of a functional value (here, $f(x) = \pi r(x)^2$) and an infinitesimal change along an axis. Since the dx in the expression represents a factor in the product, teachers will have an answer when students ask the perennial question, "Why do I have to have a dx in my integral?" Clear discussion of the Fundamental Theorem of Calculus in combination with this development via differentials will also clarify and answer another perennial question, "Where did the dx go?", since the theorem states that we can evaluate such limited sums as a *net difference* of quantities represented by an antiderivative.

4. **Don't neglect the potential of visual representations and real world modeling in your teaching.**

It is easy to treat the Fundamental Theorem of Calculus as a magic machine for symbolic manipulation, in effect as a definition and not a theorem. The difficult process of limited Riemann sums is magically changed into a simple difference of values of an antiderivative. However, such an approach generally leaves students vulnerable to the misconceptions mentioned above and a host of others that they can create for themselves. The introductory example given above shows how a real-world example can be used both to entice students into thinking about the Fundamental Theorem, and also as a way to develop the fundamental ideas of the theorem and its reasonableness. Such examples also give students a chance to develop a strong sense of the meanings that visual representations can carry. As new applications are developed and new real-world situations are modeled, a student's understanding of such visual concepts as "area" can be broadened to include "total number of bytes," "distance traveled," "length of arc," "volume of water in a pool," "work done by a pulley," *etc.*

Certainly students need to develop skill in evaluation of definite integrals via antiderivatives and the Fundamental Theorem of Calculus. It is important, however, that these not be divorced from their meaning, either through the omission of visual representations, or through carelessness with realities such as units.

5. **Clear up misconceptions associated with the other version of the Fundamental Theorem.**

In a study of students' understanding of the Fundamental Theorem of Calculus, Thompson (1994) asked nineteen students, "What letter goes in the blank to define this function: $F(\_\_) = \int_a^x f(t)dt$?" Sixteen said that t goes in the blank.

This is a common misconception and demonstrates that students do not have a good understanding that this integral does in fact define a function. The following activity may support and strengthen students' understanding that this integral is a function of x.

Give students a constant (or linear) function such as $f(t) = 2$. Have them make a table of values $(x, F(x))$ for $F(x) = \int_a^x f(t)dt$ by determining the area under the graph of $f(t)$ from $t = a$ to $t = x$ (it may be of value to have different students use different values of a). The students can plot these points and construct a graph of $F(x)$. This would be a good time to have a discussion about the function $F(x)$, what it represents, and how it relates to $f(t)$. (A dynamic demonstration of this relationship can be found at the following webpage: http://www.calculusapplets.com/accumulation.html.)

This same graph can be used to show that $\int_a^b f(x)dx = F(b) - F(a)$. Students can compare the value of $F(b) - F(a)$ obtained with the antiderivative and the value of $F(b) - F(a)$ obtained from the graph of $F(x)$ they just constructed. It may be of some value to have students show this special relationship using different values of a and b.

References

[1] M.P. Carlson, N. Smith, and J. Persson. Developing and connecting calculus students' notation of rate of change and accumulation: the fundamental theorem of calculus. International Group for the Psychology of Mathematics Education, 2003.

[2] P.W. Thompson. Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26(2):229–274, 1994.