

Report on “Inconsistencies in the Learning of Calculus and Analysis” by David Tall

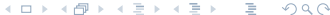
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Mapping the Calculus Curriculum, April 2009



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- Mind
- Mathematics
- Message
 - Use of language
 - Presentation of ideas

Inconsistency in students' mental conceptions

- *If you know the definition of the limit of a sequence, write it down: $s_n \rightarrow s$ as $n \rightarrow \infty$ means:*
[10/36]
- *Is $0.\bar{9}$ (nought point nine recurring) equal to one, or is it just less than one? Explain the reason behind your answer.*
[1/36]
- Thirteen thought $0.\bar{9} < 1$ while saying

$$\lim_{n \rightarrow \infty} \left(1 + \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} \right) = 2$$



Inconsistency in the belief structures in mathematics

... a branch of higher mathematics that deals with variable, or changing, quantities ... based on the concept of infinitesimals (exceedingly small quantities) and on the concept of limits (quantities that can be approached more and more closely but never reached). — Encyclopedia of Mathematics, 1982.



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Conflict between mind and mathematics

Concept Image

... the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.

Concept definition

... a form of words used to specify that concept



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Students' Concept Image of Continuity

Which of the following functions are continuous?

If possible, give reason for your answer.

$$f_1(x) = x^2$$



$$f_2(x) = 1/x \quad (x \neq 0)$$



$$f_3(x) = \begin{cases} 0 & (x \leq 0) \\ x & (x \geq 0) \end{cases}$$



$$f_4(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$$



$$f_5(x) = \begin{cases} 0 & (\text{rational}) \\ 1 & (\text{irrational}) \end{cases}$$

	C	DC	NR
f_1	41	0	0
f_2	6	35	0
f_3	27	12	2
f_4	1	38	2
f_5	8	26	7

Figure: Concept image of continuity

As $B \rightarrow A$ the line
through AB tends
to the tangent AT .

True/False ?

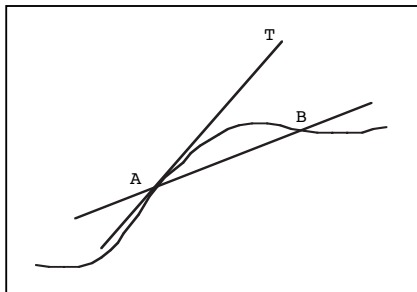


Figure: Tending to the tangent

The Message: Language

Student responses to the question “What is a function?”

- An equation with a variable factor—tells us what happens to a variable factor. e.g. $f(x) = x + 2$.
- A process which can be performed on any number and is represented in algebraic form using x as a variable.
- An order which plots a curve or straight line on a graph.
- A series of calculations to determine a final answer.
- A term by which a sequence of numbers can be written and values calculated.
- A set of instructions that you can put numbers through.



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Difference between functions and equations

Is $y = 4$ a function?



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The Message: Sequencing

Incorrect beliefs arise from starting with polynomial calculus

- that a function must be given by a formula (and only one formula is allowed)
- that every function is differentiable, except possibly at a few isolated points
- that the graph of a function looks fairly smooth with reasonably shaped maxima and minima
- that graphs always have tangents
- that a tangent touches the curve at one point only and does not cross the graph, etc etc.



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Using Technology to Reduce Cognitive Conflict

There are two possible ways of coping with conflict (which are not mutually exclusive): one is to research the cognitive conflict to be prepared to face it when it occurs, a second is to give a richer conceptualization from the start to reduce the later conflict, or at least give the experience to set it in context.