The Chain Rule

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Conceptual Difficulties with the Chain Rule

- Physically a complex idea, coordinating three or more changing quantities.
- Executing the chain rule algebraicalliy requires sophisticated abilities in recognizing algebraic form.
- Making the translation between the physical and algebraic manifestations is difficult.

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Coordination of Skill and Understanding

The progression from evaluating at numbers to finding the formula for the derivative.

- Start with the "variable form" of the chain rule, situations where you have actual numbers, so never have to explicitly substitute the inside function:

$$\left.\frac{dy}{dx}\right|_{t=3} = \left.\frac{dy}{du}\right|_{u=5} \left.\frac{du}{dx}\right|_{x=3}$$

- Move to symbolic forms where $u = g(x)$.
- Learn to see $f'(g(x))g'(x)$ "all at once" (good preparation for integration).

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Grammar

- High level symbol use
  - $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
  - $$D(f \circ g) = ((Df) \circ g)Dg$$
  - $$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$
- This is the first place where you have to say the derivative with respect to something.
- Inside and outside function.
- Systematic use of the word *argument* might be helpful.
  - If the argument of sin changes by 2π then the value does not change. What change in x makes the argument of sin change by 2π in the function $\sin(3x + 5)$.
  - $\sin(3x + 5)$ changes from 0.48 to 0.72 as the argument of sin changes from 0.5 to 0.8.

Transporting knowledge

- Recognizing situations where the chain rule applies involves identifying dependences between more than two quantities and chaining them together.
- Translating this into symbolic form is a complex act of symbolic reasoning.

Repeated Reasoning

- Understanding facilitates the use of skill: e^{x^2} looks different from $\sin(x^2)$ looks different from $\sqrt{x^2+5}$.
- Dynamic view of expressions, successively zooming in and forgetting about the outer.

Justification for the chain rule

- Right way to think of this is

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x}\frac{\Delta x}{\Delta t}$$

- Important to have a notion of informal proof.

A plane flying horizontally at an altitude of 3 miles and a speed of 600 mi/hr passes directly over a radar station. When the plane is 5 miles away from the station, at what rate is the distance from the plane to the station increasing?

A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must the air be removed when the radius is 9 cm?

Coffee is poured at a uniform rate of 20 cm3/sec into a cup whose inside is shaped like a truncated cone. If the upper and lower radii of the cup are 4 cm and 2 cm, respectively, and the height of the cup is 6 cm, how fast will the coffee level be rising when the coffee is halfway up the cup?