Limits & Continuity

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Conceptual Difficulties

- $\exists$ several disconnected notions of limits.
- Notation used in Riemann sums is confusing.
- "Limit" has both a technical and a common meaning.
- Full definition of continuity not "necessary"?

Examples

$$\lim_{x\to 2} x^2 \qquad \lim_{x\to\pi} e^x$$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \qquad \lim_{x\to\infty} \frac{x^3-3x^2}{x^2+3}$$

$$\lim_{x\to\infty} xe^{-x} \qquad \lim_{n\to\infty} \int_0^n xe^{-x}\,dx$$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \qquad \lim_{x\to\infty} \sin x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(1+\frac{i}{n}\right)^2 \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(1-a)^i}{i}$$

$$-\lim_{x\to 0} \frac{(a+x)\ln(a+x)-(a+x)-a\ln a+a}{x}$$

Examples

Use the definition of limit to investigate the behavior of the functions below and their derivatives near $x = 0$.

$$f(x) = x\sin(1/x) \qquad g(x) = x^2\sin(1/x)$$

Limits

- Algebraic computation of limits is disconnected from derivatives.

$$\frac{df}{dx} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \qquad \text{vs.} \qquad \frac{d}{dx}x^n = nx^{n-1}$$

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

Numerically? Graphically?
Geometrically? L'Hôpital?

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- Limits used in defining derivatives are disconnected from limits used in defining integrals.

$$\lim_{x \to a} \qquad \text{vs.} \qquad \lim_{n \to \infty}$$

Riemann sums

- First use of subscripted variables
- First use of summation notation

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

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Can this be resolved using "statistics" notation?

$$\sum f(x)\Delta x$$

Definition of limit

- *Further expansion of the town was limited by the river.* (limited = bounded)

$$\lim_{x \to \infty} \sin x$$

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