From Arithmetic to Algebra

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Introduction.

This note explores the boundary between arithmetic and algebra. It presents problems that run the gamut from very simple single step problems that students might meet in first or second grade, through more or less general systems of linear equations in two independent variables. For each problem, two solutions are given.* One solution is arithmetic, in the sense that it uses no variables, but simply makes a succession of calculations based on the information in the problem. The other is algebraic. It defines variables, creates expressions and formulates equations in those variables. Then it solves the equations using the standard moves of elementary algebra.**

I hope that seeing material like this collection could help students learn algebra. Three points that might be valuable for students to appreciate are:

i) that all the problems, from the simplest to the more challenging ones, are amenable to both approaches;

ii) that the two approaches are strongly parallel, in the sense that the actual sequence of calculations is the same in both solutions; and

iii) that as the problems progress, the arithmetic approach requires more and more effort, while the algebraic approach remains at more or less the same level of difficulty.

Point iii) means that, while in the earlier problems using algebra might seem to be pedantic overkill, in the later problems algebra may well provide the easier approach. The three points together support the hope that study of these problems or similar problem collections can help students become comfortable with algebra, and in particular, to appreciate that, far from being radically different subjects, arithmetic and algebra are closely related, and more precisely, that algebra is a sort of big brother of arithmetic, that facilitates solutions of problems that might seem forbidding to approach using unaided arithmetic.

Some other points may be worth noting.

iv) The algebraic solutions allow more flexibility. While we have solved them in such a way as to make the operations parallel to the arithmetic solution, once the equations are written, there are many alternate routes one could take to finding the solution, These routes should have parallel arithmetic solutions, but the arithmetic solutions may seem quite different from each other, and hard to convert from one to the other, while the algebraic approach provides a context from which alternate solutions are easily compared. In this sense, algebra better displays the problem structure than does arithmetic.

Conversion between different algebraic solutions may well involve important principles. Especially, commutativity and associativity of addition and multiplication, and the other Rules of Arithmetic may be required to justify manipulations needed to compare solutions. Studying alternate solutions could thus afford opportunities to make explicit the role of the Rules of Arithmetic.

v) We have not provided any commentary for the solutions. Probably both solutions, especially of the later problems, would need considerable class discussion in order to be absorbed. In the algebraic solutions,

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* For problem #10, a third solution is given. We leave the explanation of this solution as an exercise for the reader. This could make a very interesting class discussion.

** The main standard moves are:

i) adding or subtracting a given quantity from both sides of an equation. (Often, the quantity will be a term that appears on one side of the equation. After the operation, the same term will no longer appear on the side where it was, and will reappear on the other side of the equation, but with the opposite sign. Sometimes this special case is taught in abbreviated form without anchoring it in the general principle of "Equals added to equals make equals.")

ii) using the Distributive Rule to combine two terms with a common factor.

These two operations are roughly what is being referred to by the phrases "al jabr" and "al muqaballa" that appeared in the title "Kitab al-jabr wa-l-muqabala" of the first treatise on (what we now call) algebra, written in 820 by Al-Khwarizmi.

we assume that the reader knows the standard grammar of expressions, and is familiar with the standard manipulations involved in solving equations. For students learning algebra, dealing with a sequence of problems like these would provide occasion for discussing the formation and transformation of expressions, and the principles behind the usual moves for solving equations. This is a natural occasion for teaching the Rules of Arithmetic. In particular, the key move of "collecting like terms" is an instance of the Distributive Rule.

vi) To encourage good habits, the algebraic solutions presented follow a three part format. First, the variable or variables is carefully defined, including units. When the variable represents the number of some item, the phrase "number of" is always included in the definition. Thus, in the first problem, rather than saying, "Let t be trucks.", it is specified that t is the number of trucks. Care in defining variables can help students avoid confusion in more complex situations.

vii) Since it is normal in the US to solve some of the later problems with algebra, some of the ideas used to solve them arithmetically may not be as familiar as the algebraic methods. One method that can be quite useful is the *method of false position.* This consists in guessing a solution, then adjusting your guess according to the failure to satisfy the equation. The method of false position has the general virtue of reducing a general first order equation ($y = ax + b$, aka *affine equation*) to a simple proportion ($y = ax$, aka *(homogeneous) linear equation*). Clever "guessing" can help keep the computations needed as simple as possible. See problems # 9, # 13, and # 14.

viii) In learning to write equations to describe a situation, attention should be paid to units. In particular, it should be checked that each term of an equation is denominated in the same units, so that the units can be factored out, resulting in a purely numerical equation.

There are some further comments about particular problems, and about the method of false position, at the end.

The probems

1. One-Step Equation, A.

J. had some model trucks.
Today, he bought four more.
Now he has seven trucks.
How many trucks did he have?

Arithmetic solution:

If J. bought four trucks,
and ended up with seven trucks,
before he bought the new trucks,
he had 7 - 4 = 3 trucks.

Algebraic solution:

Defining the variable.

Let t be the number of trucks J. had.

Writing the equation.

$$t + 4 = 7.$$

Solving.

$$t + 4 = 7 \quad \Rightarrow \quad (t + 4) - 4 = 7 - 4,$$

or

$$t = 3.$$

2. One-Step Equation, B.

J. bought three packs of balloons.
He opened them and counted 12 balloons.
How many balloons are in a pack?

Arithmetic solution:

If three packs have twelve balloons,
then one pack has $12 \div 3 = 4$ balloons.

Algebraic solution:

Defining the variable.

Let b be the number of balloons in a pack.

Writing the equation.

$$3b = 12$$

Solving.

$$3b = 12 \quad \Rightarrow \quad \frac{3b}{3} = \frac{12}{3},$$

or

$$b = 4.$$

3. Two-step Equation.

J. has 4 packages of balloons and five single balloons. In all he has 21 balloons. How many balloons are in a package?

Arithmetic Solution:

Since there are 5 loose balloons, and 21 in all, there are 21 - 5 = 16 balloons in packages. Since there are 4 packages, each package holds $\frac{16}{4} = 4$ balloons.

Algebraic solution:

Defining the variable.

Let b be the number of balloons in a pack.

Writing the equation.

$$4b + 5 = 21$$

Solving.

$$4b + 5 = 21 \quad \Rightarrow \quad 4b = 21 - 5 = 16,$$

$$\Rightarrow \quad \frac{4b}{4} = \frac{16}{4},$$

or

$$b = 4.$$

4. Two-groups problem.

There are 36 children in a class. There are 4 more boys than girls.
How many boys are in the class, and how many girls?

Arithmetic Solution:

Take away four boys. This leaves 36 - 4 = 32 students, and an equal number of boys and girls.
Therefore, there are $\frac{32}{2} = 16$ girls.
Since there are four more boys than girls, there are $16 + 4 = 20$ boys.

Algebraic solution:

Defining the variables.

Let b be the number of boys in the class, and let g be the number of girls.

Writing the equations.

$$b + g = 36, \quad \text{and} \quad b = g + 4.$$

Solving.

$$b + g = (g + 4) + g = 2g + 4$$

$$\Rightarrow \quad 2g + 4 = 36, \quad \Rightarrow \quad 2g = 36 - 4 = 32,$$

$$\Rightarrow g = \frac{2g}{2} = \frac{32}{2} = 16.$$

Hence $b = g + 4 = 16 + 4 = 20$.

5. Balance Problem.

If a bar of soap balances $\frac{3}{4}$ of a bar of soap and $\frac{3}{4}$ of a pound,
how much does the bar of soap weigh?

Arithmetic Solution:

Since there is a full bar of soap on one side of the balance, and $\frac{3}{4}$ bar of soap on the other,
we may remove $\frac{3}{4}$ of a bar of soap from each side of the balance.
The objects remaining on each side will still balance. Thus $\frac{1}{4}$ bar of soap weighs $\frac{3}{4}$ pounds.
A full bar of soap is 4 $\frac{1}{4}$ths, so it weighs $4 \times \frac{3}{4} = 3$ pounds.

Algebraic solution:

Defining the variable.

Let S be the weight of the bar of soap, in pounds.

Writing the equation.

The weight of one bar of soap is S.

The weight of $\frac{3}{4}$ bar of soap and $\frac{3}{4}$ pounds is $\frac{3}{4}S + \frac{3}{4}$.

Hence,

$$S = \frac{3}{4}S + \frac{3}{4}.$$

Solving.

$$S = \tfrac{3}{4}S + \tfrac{3}{4},$$

$$\Rightarrow S - \tfrac{3}{4}S = \tfrac{1}{4}S = \tfrac{3}{4},$$

$$\Rightarrow S = 4 \times \tfrac{1}{4}S = 4 \times \tfrac{3}{4} = 3.$$

6. Counting Comparison Problem

A man wants to share his coins equally among some friends.
If he gives each friend 6 coins, he would have 4 coins left over.
If he gives each friend 7 coins, he would be 5 coins short.
With how many friends does he share?

Arithmetic Solution:

If he goes from giving 6 coins to giving 7 coins, he goes from having 4 extra to having 5 too few.
That is, one more coin for each friend makes him go from 4 too many to 5 too few,
a difference of 9 coins.
Therefore, he has 9 friends.

Algebraic Solution:

Defining the variable.

Let F be the number of friends the man shares with.

Writing the equation.

If he gives 6 coins to each friend, he gives out $6F$ coins.
If he has four left over, then in all he has $6F + 4$ coins.

If he gives 7 coins to each friend, he gives out $7F$ coins.
If he is 5 coins short of being able to do this, then in all he has 7F - 5 coins.

Both the expressions above describe the number of the man's coins. Therefore

$$6F + 4 = 7F - 5.$$

Solving:

$$6F + 4 = 7F - 5 \quad \Rightarrow \quad 4 = F - 5,$$
(Subtract $6F$ from both sides of the equation.)
$$\Rightarrow 4 + 5 = F, \quad \text{or} \quad F = 9.$$
(Add 5 to both sides of the equation.)

7: Two Rate Problem A (Direct).

J. drove from Boston to Washington and back.
Going to Washington, he drove at 100 $\frac{\text{km}}{\text{hour}}$.
Returning, traffic was heavier - he drove only at 90 $\frac{\text{km}}{\text{hour}}$.
It is 900 kilometers from Boston to Washington.
How long did the driving take?

Arithmetic Solution:

Going to Washington, at 100 $\frac{\text{km}}{\text{hour}}$, it would take $\frac{900}{100} = 9$ hours to travel 900 kilometers.
Returning to Boston, at 90 $\frac{\text{km}}{\text{hour}}$, it would take $\frac{900}{90} = 10$ hours to travel 900 kilometers.
Therefore, the total driving time is 9 + 10 = 19 hours.

Algebraic solution:

Defining the variable.

Let T be the total driving time.
Let G be the time going to Washington, and let R be the time returning.

Writing the equation.

At 100 kph, the distance driven in G hours is 100G kilometers. Since the total distance is 900 kilometers,

$$100G = 900.$$

At 90 kph, the distance driven in R hours is 90 R kilometers. Since the total distance is 900 kilometers,

$$90R = 900.$$

Since the total driving time is the sum of the distance going and the distance returning,

$$T = G + R.$$

Solving.

$$100G = 900 \qquad \Rightarrow \qquad G = 9.$$

$$90R = 900 \qquad \Rightarrow \qquad R = 10.$$

Hence

$$T = G + R = 9 + 10 = 19.$$

8. Catch Up Problem.

H. and J. are gaining weight for football.
H. weighs 205 pounds and is gaining 2 pounds per week.
J. weighs 195 pounds and is gaining 3 pounds per week.
When will they weigh the same?

Arithmetic solution:

J. starts 205 - 195 = 10 pounds lighter.
J. catches up at 3 - 2 = 1 pounds per week.
So it takes $10 \div 1 = 10$ weeks to catch up.

Algebraic solution:

Defining the variable.

Let w be the number of weeks until they weigh the same.

Writing the equation.

H's weight after w weeks is $205 + 2w$.
J's weight after w weeks is $195 + 3w$.

When they are equal,

$$195 + 3w = 205 + 2w$$

Solving.

$$195 + 3w = 205 + 2w$$

$$\Rightarrow \qquad 3w = 205 + 2w - 195 = 2w + 10,$$

$$\Rightarrow \qquad 3w - 2w = 10, \qquad \text{or} \qquad w = 10.$$

9. Two-price Problem.

Tickets for the class show are \$3 for students, and \$10 for adults.
The auditorium holds 450 people.
The show was sold out, and the class raised \$2750 in ticket sales.
How many students bought tickets?

Arithmetic Solution:

The class sold 450 tickets. If all the tickets had been for adults, total sales would have been \$4500.

Instead, the sales were \$2750, which is \$4500 - \$2750 = \$1750 less.

Since each student ticket brings in \$10 - \$3 = \$7 less than an adult ticket,
there must have been

$$\frac{1750}{7} = 250$$

student tickets sold.

Algebraic solution:

Defining the variables.

Let S be the number of student ticket sold,
and let A be the number of adult tickets sold.

Writing the equations.

Number of tickets sold:

$$A + S = 450.$$

Dollar Value of Tickets Sold:

$$10A + 3S = 2750.$$

Solving.

$$10(A + S) - (3S + 10A) = 4500 - 2750,$$

$$\Rightarrow \quad 7S = 1750 \quad \Rightarrow S = 250.$$

10. Mixture Problem, Open Bucket.

Given 10 gallons of 3% vinegar, how much 9% vinegar should be added to make a 5% mixture?

Arithmetic solution:

If we have 10 gallons at 3%, we have .3 gallons of (pure) vinegar.
To have 10 gallons of 5%, we need .5 gallons of (pure) vinegar.
So we are .2 gallons short.

If we have a gallon of 9% vinegar, we have .09 gallons (pure) vinegar.
This is .04 gallons more than a 5% solution.
So for each gallon added of 9%, we gain .04 gallons over a 5% solution.

Since we need to gain .2 gallons, we should add $\frac{.2}{.04} = 5$ gallons.

Algebraic solution:

Defining the variable.

Let g be the number of gallons of 9% vinegar to be added.

Writing the equation.

Total vinegar in 10 gallons at 3% = .3 gallons.

Total vinegar in g gallons at 9% = $.09g$ gallons.

Vinegar in (10 + g) gallons at 5% = $.05(10 + g)$.

When the mixture is 5% vinegar,we have

$$.3 + .09g = .05(10 + g).$$

Solving.

$$.3 + .09g = .05(10 + g) = .5 + .05g$$

$$\Rightarrow .09g = .5 + .05g - .3 = .05g + .2$$

$$\Rightarrow .09g - .05g = .2, \qquad \text{or} \quad .04g = .2$$

$$\Rightarrow \quad g = \frac{.5}{.04} = 5.$$

Proportional Solution:

We have a 3% solution. We want a 5% solution. We will add a 9% solution.

Since 5 is twice as close to 3 as it is to 9 (that is, 9 - 5 = 2(5 - 3)), we should add only half as much 9% as we have 3% to start with.

Thus, we should add $\frac{10}{2} = 5$ gallons.

11. Two Rate Problem B.

Mei Lin paints a room in 4 hours, and Mei Hua paints a room in 3 hours. How long will it take them to paint a room working together?

Remark: In problems like this, the unstated assumptions should be discussed. The assignment of such problems might include, to state the assumptions one needs to be able to solve the problem.

Arithmetic Solution:

Since Mei Lin paints a room in 3 hours, she paints $\frac{1}{3}$ of a room in one hour.
Since Mei Hua paints a room in in 4 hours, she paints $\frac{1}{4}$ of a room in one hour.
So together, they paint $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of a room in one hour.
Therefore, it will take them $\frac{12}{7}$ of an hour, or about 1 hour, 43 minutes.
to paint the room together.

Algebraic solution:

Defining the variable.

Let T be the time in hours for painting the room together.

Writing the equation.

Since Mei Lin paints one room in 3 hours, she will paint $\frac{T}{3}$ of a room in T hours.
Since Mei Hua paints one room in 4 hours, she will paint $\frac{T}{4}$ of a room in T hours.

Since they paint one room in T hours,

$$\frac{T}{3} + \frac{T}{4} = 1.$$

Solving.

$$\frac{T}{3} + \frac{T}{4} = (\frac{1}{3} + \frac{1}{4})T = (\frac{7}{12})T = 1.$$

Hence,

$$T = \frac{12}{7} \times ((\frac{7}{12})T) = \frac{12}{7} \times 1 = \frac{12}{7}.$$

12. General 2× 2 Problem.

Eng-Chye bought 3 notebooks and 7 pencils for \$4.
Chen-bo bought 5 notebooks and 5 pencils for \$5.
What is the price of a notebook?

Arithmetic Solution:

If Eng-Chye buys 5 sets of 3 notebooks and 7 pencils, he will pay \$20 for 15 notebooks and 35 pencils.
If Chen-bo buys 7 sets of 5 notebooks and 5 pencils, he will pay \$35 for 35 notebooks and 35 pencils.
The difference between the two purchases is 20 notebooks, and the difference in price is \$15.
So 20 notebooks cost \$15, or one notebook costs \$ $\frac{15}{20} = \$.75$.

Algebraic solution:

Defining the variables.

Let N be the price of a notebook, in dollars, and let P be the price of a pencil, in dollars.

Writing the equation.

Eng-Chye's purchase: 3 N + 7 P = 4.

Chen-bo's purchase: 5 N + 5 P = 5.

Solving.

$$7(5N + 5P) = 35N + 35P = 7 \cdot 5 = 35.$$

$$5(3N + 5P) = 15N + 35P = 5 \cdot 4 = 20.$$

Subtracting the second equation from the first gives

$$20N = 15, \quad \text{or} \quad N = \frac{15}{20} = .75.$$

13. Two Rate Problem C (Indirect version of A (# 8)).

J. drove from Boston to Washington and back.
Going to Washington, he drove at 100 $\frac{\text{km}}{\text{hour}}$.
Returning, traffic was heavier - he drove only at 90 $\frac{\text{km}}{\text{hour}}$.
The total driving time was 19 hours.
How far is it from Boston to Washington?

Arithmetic Solution:

Let's guess the distance. Suppose it is 100 km.
Then it would take us $\frac{100}{100} = 1$ hour, to drive down, and $\frac{100}{90} = 1\frac{1}{9}$ hour to drive back.
So the whole trip would be $1 + 1\frac{1}{9} = 2\frac{1}{9} = \frac{19}{9}$hours.
The actual trip took 19 hours, which is 9 times as long, so the actual distance must be 9 times our guess, or $9 \times 100 = 900$ kilometers.

Algebraic solution:

Defining the variable.

Let D be the distance from Boston to Washington, measured in kilometers.

Writing the equation.

The number of hours driving to Washington is $\frac{D}{100}$.

The number of hours returning to Boston is $\frac{D}{90}$.

Hence, the total driving time is $\frac{D}{100} + \frac{D}{90}$, so

$$\frac{D}{100} + \frac{D}{90} = 19.$$

Solving.

$$\frac{D}{100} + \frac{D}{90} = D(\frac{1}{100} + \frac{1}{90}) = D(\frac{90}{100\cdot 90} + \frac{90}{100\cdot 90}) = D(\frac{190}{100\cdot 90}).$$
$$\Rightarrow D(\frac{190}{100\cdot 90}) = 19,$$
$$\Rightarrow D = 19 \cdot \frac{100\cdot 90}{190} = 900.$$

Appendix

14: Cows and chickens

In a farmyard, there are cows and chickens.

There are 50 heads and 120 feet.

How many cows are there? How many chickens?

Arithmetic Solution:

Chickens have two feet, and cows have four feet. Each has one head.
If there were only chickens, there would be $50 \times 2 = 100$ feet.

But there are 120 feet.
The extra 20 feet must come from cows.
Each cow has 2 more feet than a chicken, so there are $\frac{20}{2} = 10$ cows, and 50 - 10 = 40 chickens.

Algebraic solution:

Defining the variables.

Let C be the number of chickens, and let D be the number of (dairy) cows.

Writing the equations.

Number of heads:

$$C + D = 50.$$

The number of feet:

$$2C + 4D = 120.$$

Solving.

$$(2C + 4D) - 2(C + D) = 120 - 2 \times 50,$$

$$\Rightarrow \quad 2D = 20 \quad \Rightarrow D = 10.$$

Concluding Remarks

a) Although it has a different feel, the above problem is of the same type as # 9.

b) Both problems were solved by the *method of false position,* in which you propose a solution, then modify it to find the true solution.

The trial problem can be chosen to reduce the number of variables from two to one, and at the same time, to convert the problem into a simple division (one-step problem B).

c) The method of false position was also used in the arithmetic solution to problem 13. In that solution, we made a guess for the distance, then calculated the time required for the trip, which amounted to the direct problem (as in problem 7) for that distance. Then we invoked a principle of proportionality: that the ratio of the time for the actual trip to the time for the hypothetical trip is the same as the ratio of the actual distance to the hypothetical distance. Particularly in a relatively complex situation such as the one of this problem, understanding the proportionality principle requires a fair degree of sophistication on the part of the solver, probably at least at the level of the algebraic solution. Thus, although the method of false position provides a solution to problem 11 that is arithmetic in the sense that it does not explicitly use variables, one might argue that the level of sophistication required to produce such a solution is not less than what is required for the algebraic solution.

Use of the method of false position in the two-price problem or the cows and chickens problem also invokes proportionality, but at a lower level, more or less at the level of the definition of multiplication (problem 2).

d) The direct trip problem 7 is much easier than the inverse trip problem 13.

This is because, in the direct problem, one can divide the known quantity D with each of the given rates to find the time each part of the trip, and then form the sum

$$\frac{D}{r_1} + \frac{D}{r_2}$$

to find the total time.

However, to solve the inverse problem, one must somehow form the expression

$$D(\frac{1}{r_1} + \frac{1}{r_2}),$$

in order to divide by the coefficient

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2}.$$

In the algebraic solution this is accomplished using the Distributive Rule, in the form of "collecting like terms". In the arithmetic solution, it is done by stealth, by solving the inverse problem for a selected distance, and then invoking proportionality.

e) If students can solve all the problems above arithmetically, and still need to be convinced of the need for algebra, assign problems like the following.

15: Three Rate Problem.

Alex, Basu and Chiwai are going to work together to paint a room.

If Alex and Basu work together, they can paint the room in 2 hours.

If Alex and Chiwai work together, they can paint the room in 3 hours.

If Basu and Chiwai work together, they can paint the room in 4 hours.

How long will it take them to paint the room?