VIGRE Workshop on Teaching Proof

Steven Gagola Daniel Madden William McCallum
Douglas Ulmer

Institute for Mathematics and Education

September 2008

The problem (from Barbara Shipman)

Some of my students in Analysis I still have trouble with properly declaring the existence of, stating the meaning of, or definining objects used in the proof. For example, I have seen mistakes such as the following:

1. Given that a function g from X to Y is surjective, a student might write "If y is in Y, then $g(x) = y$". The reader is left to wonder what x is.
2. Sometimes a student will define a function h from $\mathbb{N}$ to $\mathbb{N}$ as something like $h(x) = f(x/2)$ if x is even and $h(x) = g((x+1)/2)$ if x is odd, without telling the reader what f and g are.
3. A student may start a proof by writing "For all $\epsilon > 0$, there exists $\delta > 0$ such that ..." and immediately thereafter consider the interval $(f(p) - \epsilon, f(p) + \epsilon)$, without having chosen any particular ϵ.

A possible solution

This series of problems is intended to lead students to grapple with the different contexts in which the same variable is used.

1. Find a value of x which minimizes $f(x) = x^2 - 2x + 2$.
2. Prove that $f(x)$ is greater than zero for all values of x.
3. Prove that there exists an x such that $f(x)$ is twice the minimum value of $f(x)$.
4. Prove that there does not exist an x such that $f(x) = -1$.
5. Prove that the set of matrices of the form

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, \quad x, y \in \mathbb{R}$$

   is closed under matrix multiplication.
6. Show that the multiplication in (5) is not commutative.