The problem (from Barbara Shipman)

Some of my students in Analysis I still have trouble with properly declaring the existence of, stating the meaning of, or defining objects used in the proof. For example, I have seen mistakes such as the following:

1. Given that a function $g$ from $X$ to $Y$ is surjective, a student might write “If $y$ is in $Y$, then $g(x) = y$”. The reader is left to wonder what $x$ is.

2. Sometimes a student will define a function $h$ from $\mathbb{N}$ to $\mathbb{N}$ as something like $h(x) = f(x/2)$ if $x$ is even and $h(x) = g((x + 1)/2)$ if $x$ is odd, without telling the reader what $f$ and $g$ are.

3. A student may start a proof by writing “For all $\epsilon > 0$, there exists $\delta > 0$ such that . . . ” and immediately thereafter consider the interval $(f(p) - \epsilon, f(p) + \epsilon)$, without having chosen any particular $\epsilon$. 
A possible solution

This series of problems is intended to lead students to grapple with the different contexts in which the same variable is used.

1. Find a value of \( x \) which minimizes \( f(x) = x^2 - 2x + 2 \).
2. Prove that \( f(x) \) is greater than zero for all values of \( x \).
3. Prove that there exists an \( x \) such that \( f(x) \) is twice the minimum value of \( f(x) \).
4. Prove that there does not exist an \( x \) such that \( f(x) = -1 \).
5. Prove that the set of matrices of the form

\[
\begin{pmatrix}
    x & y \\
    0 & 1
\end{pmatrix}, \quad x, y \in \mathbb{R}
\]

is closed under matrix multiplication.
6. Show that the multiplication in (5) is not commutative.