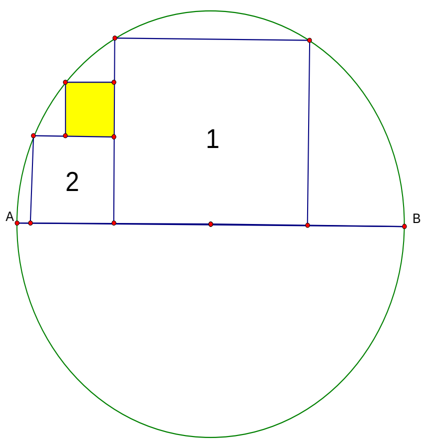
Sample Exam C
Math in the Middle – Master of Arts in Teaching (MAT) Masters Exam – Summer 2006
UNL Department of Mathematics

The MAT exam consists of a (two-part) written section and an oral presentation to an MAT faculty committee. In order to provide sufficient time for grading Part IA and to give you as much time as possible to complete the paper you are writing for Part IB, we are requesting that the written part of your exam be submitted in two parts. Part IA is to be submitted on or before Monday, July 17, 2006. Part IB is to be submitted on or before Monday, July 24, 2006. Math in the Middle oral presentations are scheduled for Thursday and Friday, July 27&28, 2006.

Part IA. Below find three questions that your Math in the Middle coursework should prepare you to answer. It is anticipated that taken together, your answers will be about 5 to 8 pages.

1. Math 802T-2 – In the unit circle below, AB is a diameter of the circle, square 1 has one side on AB and the opposite corners on the circle, and square 2 is inscribed between AB square 1 and the circle. What is the area of the shaded square that is inscribed between square 1, and square 2, and the circle as shown?



2. Math 805T-2 – Here are the definitions of some special sequences:

Constant: $S_n = a$ for n = 0, 1, 2, 3, …, where a is constant for all n.

Linear: $S_n = an + b$ for n = 0, 1, 2, …, where a and b are constant for all n.

Quadratic: $S_n = an^2 + bn + c$ for n = 0, 1, 2, …, where a, b, and c are constant for all n.

Exponential: $S_n = ab^n$ for n = 0, 1, 2, …, where $a \neq 0$ and $b > 0$ are constant for all n.

Define the "difference sequence" Δ_n for a given sequence S_n as follows: $\Delta_n = S_{n+1} - S_n$ for n = 0, 1, 2, … . Verify:

a. If S_n is linear, then its difference sequence is a constant sequence for all n. What is the constant?

b. If S_n is quadratic, then the difference sequence of its difference sequence (the so-called 2nd difference sequence) is a constant sequence. What is the constant?

c. If S_n is exponential then both the sequence S_{n+1}/S_n and the sequence Δ_n/S_n for n = 0, 1, 2, … are constant sequences. (Here Δ_n is the difference sequence for S_n.)

d. Below are sequences A_n, B_n, C_n, and D_n. Use your work in answering parts a, b and c, determine whether each sequence is linear, quadratic, exponential, or neither. Where appropriate, find the exact values for *a*, *b*, and *c*. (You may check your work with your graphing calculator or with computer software.)

n	A_n	B_n	C_n	D_n
0	22	5	11	0.2500
1	27.0878	7.15	7.79	0.3100
2	32.1724	9.6	4.58	0.3844
3	37.2240	12.35	1.37	0.4767
4	42.2304	15.4	-1.84	0.5911
5	47.2805	18.75	-5.05	0.7329
6	52.4093	22.4	-8.26	0.9088
7	57.5466	26.35	-11.47	1.1269
8	62.6215	30.6	-14.68	1.3974
9	67.6492	35.15	-17.89	1.7327
10	72.6761	40	-21.1	2.1486

3. Discrete mathematics is not a content strand in the Nebraska State Standards or in the National Council of Teachers of Mathematics Standards. Discuss which discrete math topics that you could introduce at your grade level and explain how these topics would be integrated into your current curriculum. Would you be able to use discrete mathematics topics to achieve current curriculum goals, or will anything you add to the curriculum be an “add on?” Discuss the impact that integrating discrete math into your curriculum will have on your students.

Part IB. For this part of the exam, you should research the topic below and write an expository paper (including proofs or examples where appropriate). It is anticipated that your paper will be approximately 8 to 10 pages.

Farey Sequences, Ford Circles, Pick's Theorem – Person B

This is your opportunity to do something really different! This MAT paper has been set up as a joint project for a team of three people. (The other two teachers are Julane Amen and Anne Schmidt.) Since there is some serious mathematics here, with much of the work left to the student, we don't expect that the combined paper will be three times as long as an individual paper. Only two or three pages longer is acceptable. David Fowler has posted images related to this paper at http://www.unl.edu/tcweb/fowler/FareyFordPickImages/.

Overview

For this paper, you will explore three topics with the following interesting properties:

1. They can be explained at the level of middle school mathematics.

2. They each have applications in research-level mathematics or physics.
3. Between middle school and post-doctoral school, not much seems to have been done with them. (The connections between these three topics are not usually explored in school mathematics.)

Person A will take Farey Sequences, Person B will take Ford Circles, and Person C will take Pick's Theorem. After you complete your individual portions of the project and explain your results to each other, then as a team you will show:

1. A connection between Farey Sequences and Ford Circles.
2. The use of Pick's Theorem to prove a property of Farey Sequences.
3. A third connection of your choice.

Common features of each individual section

1. Whether your topic is Farey, Ford, or Pick, you should have some historical information on the person.
2. You should be able to give a simple example of the theorem associated with each person, and then a more general statement.
3. If a proof is possible, or an outline of a good mathematical explanation, you should include that.
4. To get the most out of this paper, try to proceed with minimal help at first, once you know a basic definition. See what you can figure out on your own. When you're really stuck, talk to a team member. When the whole team is stuck, and nobody else is around, then use your research skills to learn more about the topic.

Comments specific to the Farey section of the paper:

1. Here is a Farey sequence (also called a "Farey series" or "The Farey fractions") for n = 5:

$$\{0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\}$$

How would you describe this sequence? What do you think a Farey sequence looks like for a different small n? How do these sequences resemble what an elementary teacher might call "The Fractions?" How would you describe the general case of this sequence, using the term, "rational numbers?" Note: There is an exercise about Farey fractions in *The Heart of Mathematics*.

2. Given the numbers $\frac{1}{3}$ and $\frac{1}{2}$, a blunder every teacher has seen is $\frac{1}{3} + \frac{1}{2} = \frac{(1+1)}{(3+2)}$.

But what do you notice about the result of this "incorrect addition" when working with Farey fractions? Does this happen for other fractions in a Farey sequence?

3. Subtract 2/3 from 3/4 (correctly!). What is the numerator in the resulting fraction? Try another consecutive pair. Make and test a conjecture.

Comments specific to the Pick section of the paper:

Let A be the area of a simply closed lattice polygon. Let B denote the number of lattice points on the polygon edges and I the number of points in the interior of the polygon. Then A = I + B/2 - 1.

Verify this formula for several kinds of polygon on a geoboard.

Comments specific to the Ford section of the paper:

1. What is the relationship between the radius of the small circle and the larger circle in the diagram?
2. Can you find a formula to continue putting in small tangent circles as shown in the diagram?

Connections

A. Farey property proved with Pick.

Show that if a/b and c/d are consecutive Farey fractions, then ad - bc = 1. (Compare with a conjecture above). Here are steps you might follow:

a. Plot point (a,b) and (c,d) on a lattice.
b. Show that the area of a triangle with vertices (0, 0), (a, b) and (c,d) is given by (ad - bc)/2
c. Use Pick's Theorem to show that such a triangle will have area = 1/2.

(Note: this is essentially Coxeter by way of Penrose "Mathematical Intelligence" in *What is Intelligence?* Edited by Jean Khalfa, Cambridge Press)

B. Ford Circles determined by Farey Fractions

Show the relationship between the Farey Fractions and the Ford Circles, using a geometric diagram and the algebraic rule for the radius and center of a Ford Circle.

C. Another Connection?

Can you think of another possible connection? Maybe a generalization? Boldly go into a higher dimension? (Not for the faint-of-heart.)

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Part II. For the oral part of your exam, you should prepare a presentation based on your paper in Part IB. Your talk should also discuss your journey through Math in the Middle, emphasizing one aspect of your Math in the Middle experience:

- mathematics
- pedagogical preparation
- action research
- leadership development

The presentation should be approximately 20 minutes in length. Following your presentation, expect the committee to be interested in your work and your experiences in Math in the Middle. Thus, you should expect them to ask a few questions that raise interesting points about what you have presented. For example, the committee may ask questions designed to probe your depth of understanding of the topic about which you have written and the committee may ask questions related to the answers you provided for Part IA. In the past, most MAT candidates use a computer presentation program such as Power Point, although a prepared set of overhead transparencies or a handout is acceptable. Don't overload your slides with too many words and pictures.