4 Combinations and Locks

Introduction

Combinatorics, sometimes called "the art of counting without counting" [2], is an active branch of mathematics that has made some inroads into the high school curriculum. Most of these inroads have been in discrete mathematics courses, probability courses, or parts of topics courses. This chapter has some of these kinds of examples—how combinatorics itself can fit into the high school program—but it is also about another use of combinatorial ideas: we want to look at how combinatorics and combinatorial thinking can be used to illuminate ideas from more mainstream high school courses, like algebra.

Combinatorics provides techniques that allow you to figure out how many ways there are to do something without having to make a list of all the ways. For example, if you have three shirts and two pair of jeans, you can make 3×2 outfits. This is an example (albeit an extremely basic one) of a combinatorial result: If you have n objects in set A and m objects in set B, you can make nm ordered pairs whose first "coordinate" comes from A and whose second one comes from B. A more subtle combinatorial result was mentioned in Chapter 1 (page 26):

The set of ordered pairs (a, b) where $a \in A$ and $b \in B$ is suggestively denoted by $A \times B$.

> *The entries in Pascal's triangle count subsets.* Suppose you have a set of five elements, say $\{A, B, C, D, E\}$.

How many 3-element subsets are there? There are $\binom{5}{3} = 10$. Here they are:

$$\{A,B,C\},\{A,B,D\},\{A,B,E\},\{A,C,D\},\{A,C,E\},$$
$$\{A,D,E\},\{B,C,D\},\{B,C,E\},\{B,D,E\},\{C,D,E\}$$

You can list all the 3-element subsets of a 5-element set with very little trouble. But you'd be hard pressed to list all the 3-element subsets of a 100-element set. Even so, the result says there are $\binom{100}{3} = 161700$ of them.

Another use of combinatorics is the so-called "combinatorial proof." A typical combinatorial proof establishes some identity by showing that both sides represent different ways to perform the same counting process. For example, suppose we take the definition of $\binom{n}{k}$ to be the one above: $\binom{n}{k}$ is the number of k-element subsets of an n-element set. Suppose further that we want to show that these "subset counter" $\binom{n}{k}$ satisfy the same recurrence as the "number pattern" definition of Pascal's triangle (the rows end in 1 and any interior number is the sum of the two above it). That is, using the subset interpretation of $\binom{n}{k}$, we want to show that:

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or if } k = n \text{ (each row starts and ends with 1),} \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{if } 0 < k < n \text{ (any interior number is the sum of the two above it).} \end{cases}$$

We could argue like this:

- $\binom{n}{0} = 1$ because there's only one subset of an n-element set that has 0 elements (namely, the empty set).
- $\binom{n}{n} = 1$ because there's only one subset of an n-element set that has n elements (namely, the whole set).
- If $0 < k < n$, we can show that
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$
by a "committee" proof: Suppose you have n people and you want to form a k-member committee. The committee is just a set of k people chosen from n, so there are $\binom{n}{k}$ such committees. But let's count the number of committees in a different way: pick one of the n people, say Nina, and separately count the k-member committees that do and don't contain Nina as a member:
  1. **The committees that contain Nina.** Well, there are $k-1$ slots open, and you have $n-1$ people who can fill them, so there are
  $$\binom{n-1}{k-1}$$
  committees that contain Nina.

This "proof by committee" is an example of what Marvin Freedman, a mathematician who teaches at Boston University, calls a "story proof."

2. **The committees that do not contain Nina.** Well, there are k slots open, and you have only $n-1$ people who can fill them (the original n people, but then you exclude Nina), so there are

$$\binom{n-1}{k}$$

committees that don't contain Nina.

But every committee either contains or does not contain Nina, so there are

$$\binom{n-1}{k-1}+\binom{n-1}{k}$$

k-element committees. But there are also $\binom{n}{k}$ such committees. Hence

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}$$

as claimed.

Combinatorial proofs often allow you to establish identities (especially identities that involve binomial coefficients) that would be quite difficult and messy using algebra.

In this chapter, we'll look at several applications—of combinatorial results *and* proofs—to some topics connected to high school mathematics. More precisely,

- In section 4.1, we'll give some combinatorial proofs of some interesting identities.
- In section 4.2, we'll look at several combinatorial solutions to the famous "Simplex lock" problem.
- In section 4.4, we'll return to the problem first introduced in section 1.6 of Chapter 1: How does one move back and forth between the powers of x and the "Mahler basis" for polynomials. We'll see that the numbers that show up in these conversions also show up in one solution to the Simplex lock problem! We'll look at some reasons underneath this "coincidence."

The Simplex lock is a project in [1]. Many of the solutions in section 4.2 came from work students did in high school classes.

In addition to producing some very pretty mathematics, these topics will be used to show how the ideas of combinatorics—incisive methods for shortcutting the counting process and combinatorial proofs—can be applied to topics from the high school curriculum.

Problems

1. Suppose you take the "factorial way" to define binomial coefficients. That is, your definition is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Prove that, using this definition, $\binom{n}{k}$ is the number of k-element subsets of an n-element set.

2. Using the factorial way to define binomial coefficients, give a non-combinatorial proof of the fact that, if $0 < k < n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Get ready to do some algebra. . .

3. Give a "committee" proof of the identity $(0 \le k \le n)$:

$$\binom{n}{k} = \binom{n}{n-k}$$

If you have 20 people, picking a committee of 5 is the same as picking a non-committee of 15.

4. Give a committee proof of the identity ($2 \le k \le n-2$):

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

Pick *two* distinguished people, say, Nina and Nona.

5. Suppose you use the "subset way" of defining $\binom{n}{k}$. What would be some sensible ways of defining $\binom{6}{8}$, or in general, $\binom{n}{k}$ when $k > n$?

How many 8-element subsets of a 6-element set are there?

6. Come up with a reasonable definition of $\binom{-5}{3}$, using one of the definitions from the discussion on page 26 of Chapter 1.

7. Using your results from problems 5 and 6, write out an "expanded" Pascal's triangle that has negative rows and 0s to the right of the rightmost 1. State and explain some patterns in this expanded array.

It might help to "left justify" the triangle, the way you'd see it in a spreadsheet.

8. Let $c_{m,k}$ stand for the number of ways you can break a set of n things into k disjoint non-empty subsets. For example, $c_{3,2} = 3$, because there are three ways to break a set of three things into two non-empty subsets. Here they are:

$$\begin{array}{ccc} \{A,B\} & \cup & \{C\} \\ \{A,C\} & \cup & \{B\} \\ \{C,B\} & \cup & \{A\} \end{array}$$

Calculate a few rows of a $c_{m,k}$ triangle, "Pascal triangle style" and describe several ways to characterize the entries.

What's analogous to the "every number is the sum of the two above it" rule?

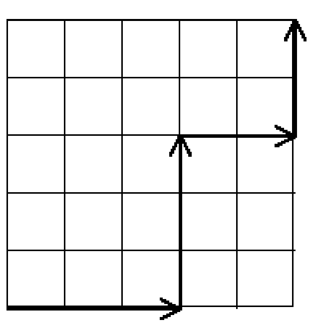
9. Let $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ stand for the number of ways you can break a set of m things into k disjoint non-empty subsets, where *it matters the order in which the subsets are listed*. For example, $\left\langle \begin{matrix} 3 \\ 2 \end{matrix} \right\rangle = 6$, because there are six ways to break a set of three things into 2-subset sequences. Here they are:

$$\begin{array}{rcl} \{A,B\} & \cup & \{C\} \\ \{A,C\} & \cup & \{B\} \\ \{C,B\} & \cup & \{A\} \\ \{C\} & \cup & \{A,B\} \\ \{B\} & \cup & \{A,C\} \\ \{A\} & \cup & \{C,B\} \end{array}$$

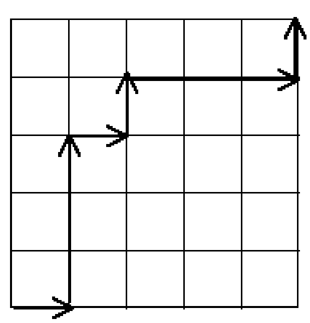
$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ is the number of *ordered* partitions of an n-element set that have k parts.

Calculate a few rows of an $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ triangle, "Pascal triangle style" and describe several ways to characterize the entries.

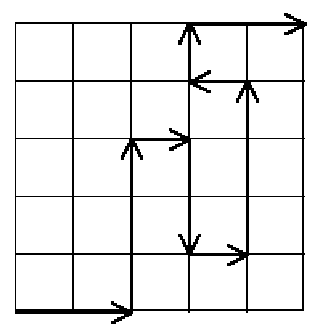
10. Ms. D'Amato likes to take a different route to work every day. She will quit her job the day she has to repeat her route. Her home and work are pictured in the grid of streets below. If she never backtracks (she only travels north or east), how many days will she work at this job?



A valid trip



Another valid trip



Not a valid trip

11. How many functions are there from a 5-element set to a 7-element set? From an m-element set to an n-element set?

12. A function is called "one-to-one" if no two elements in the domain end up at the same element of the range. How many one-to-one functions are there from a 5-element set to a 7-element set? From a 7-element set to a 5-element set? From an n-element set to an m-element set?

13. How many one-to-one functions are there from a 20-element set to a 365-element set? How many functions from a 20-element set to a 365-element set are *not* one-to-one?

14. What's the probability that at least two people in a room of 20 have the same birthday? What if there are 25 people in the room? How many people have to be in the room before the probability of two same-day birthdays is more than .5?

Hint: First find the probability that all 20 people have *different* birthdays.

15. A function is called "onto" if every element in the range gets hit by at least one element in the domain. How many onto functions are there from a 7-element set to a 5-element set? From a 5-element set to a 7-element set? From an m-element set to a k-element set?

Hint: How is this related to problem 9?

4.1 Combinatorial Proofs and Identities

In this section, we'll give a few examples of combinatorial ("story") proofs. The heart of the section is a collection of problems for you to try. Combinatorial proofs provide very simple ways to establish complicated results. In a few lines, you realize that a complex algebraic identity is, in fact, quite simple—if you look at it in the right way. The tradeoff is that finding the "right way" to look at it is notoriously difficult. The only way we know to develop the skill is to study some examples and then practice, practice, practice.

Let's begin with one of the most important results in high school algebra:

Theorem 1 (The Binomial Theorem)

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.\end{aligned}$$

Proof: There are dozens of beautiful ways to prove this theorem, but here's a combinatorial proof. Look at $(a+b)^5$. It's really just

$$(a+b)(a+b)(a+b)(a+b)(a+b).$$

Imagine doing the calculation. If you multiplied all this out, you'd get a sum of terms. You get each term by taking a letter from each parentheses and multiplying them together.

For example, you could take b from parentheses 1, 2, and 4 and a from parentheses 3 and 5. That would give you an a^2b^3. But you could also get an a^2b^3 by taking b from parentheses 1, 2, and 3 and a from parentheses 4 and 5. The *coefficient* of a^2b^3 will be all the ways you can pick 3 "b" parentheses and 2 "a" parentheses. And *that* is just the number of ways you can pick three things (three "b"s) from five parentheses. It's $\binom{5}{3}$.

Or, it's the number of ways you can pick two things (two "a"s) from five parentheses. That's $\binom{5}{2}$, which is the same as $\binom{5}{3}$.

So, you can pick no b's and five a's (that term is $\binom{5}{0}a^5b^0$), one

b and four as (that term is $\binom{5}{1}a^4b^1$), ... So,

$$\begin{aligned}(a+b)^5 &= \binom{5}{0}a^5b^0 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \binom{5}{4}ab^4 + \binom{5}{5}a^0b^5 \\ &= \sum_{k=0}^{5}\binom{5}{k}a^{5-k}b^k.\end{aligned}$$

The same idea applies in general to $(a+b)^n$. Every term will be of the form $a^{n-k}b^k$ (pick k b's and $n-k$ a's). And the *coefficient* of $a^{n-k}b^k$ is the number of ways you can pick k b's from n parentheses—it's just $\binom{n}{k}$. So the expansion will be the sum of all such terms, and

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k. \qquad \textbf{Q.E.D.}$$

This is just the beginning. By replacing a and b with particular values, you get many interesting numerical identities. For example, replace a and b by 1 and you get

Notice that this says that the sum of the entries in the nth row of Pascal's triangle is 2^n.

Corollary 1

$$\begin{aligned}2^n &= \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} \\ &= \sum_{k=0}^{n}\binom{n}{k}.\end{aligned}$$

Yes, this is easy to see by replacing a and b by 1 in the binomial theorem, but lets's look at a combinatorial proof of Corollary 1:

Ways to think about it

The right-hand side of the identity in the corollary is the sum of the number of

$$0\text{-element, 1-element, 2-element}, \ldots, n\text{-element}$$

subsets of an n-element set. So, it's the *total* number of subsets of an n-element set. That gives us the inspiration to count the total number of subsets of an n-element set in another way.

Suppose you have a set of n things, say $A = \{a_1, \ldots, a_n\}$. How many subsets does A have? Well, think about how you'd build a subset. Imagine passing over each element and deciding "yes, I want this one in my subset," or "no, I don't want this one in my subset." You want to count every possible combination of strings of length n composed of "yes" and "no."

So, all "yes"s corresponds to the whole set, all "no"s corresponds to the empty set, all strings that have two "yes"s and the rest "no"s correspond to all the 2-element sets, and so on.

But there are two choices of a_1 (yes or no), two choices for a_2, two choices for a_3, ..., two choices for a_n. In other words there are

$$\underbrace{2 \times 2 \times 2 \times 2 \ldots \times 2}_{n \text{ times}} = 2^n$$

strings of "yes" and "no," so there are 2^n possible subsets. Hence

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

because both sides count the total number of subsets of an n-element set.

Many teachers are always on the lookout for interesting identities involving entries in Pascal's triangle. One source of inspiration is to substitute well-chosen numbers into the binomial theorem (think about what you get if you replace a by 1 and b by -1, for example). Combinatorics offers another treasure house of identities: those that come from thinking about different ways to count the same thing. For example, suppose you want to pick five things from a set of twelve. There are $\binom{12}{5}$ ways to do this. But let's make up a different way to count. Suppose you split your 12-element set into two pieces, say a piece A with seven things and a piece B with five things. Now you could pick your five things by taking some from A and some from B. More precisely, you could take:

We chose 7 and 5 arbitrarily, any two non-negative integers that sum to 12 would work.

None from A and five from B: This can be done in

$$\binom{7}{0}\binom{5}{5} \qquad \text{ways}$$

One from A and four from B: This can be done in

$$\binom{7}{1}\binom{5}{4} \qquad \text{ways}$$

Two from A and three from B: This can be done in

$$\binom{7}{2}\binom{5}{3} \qquad \text{ways}$$

Three from A and two from B: This can be done in

$$\binom{7}{3}\binom{5}{2} \qquad \text{ways}$$

Four from A and one from B: This can be done in

$$\binom{7}{4}\binom{5}{1} \qquad \text{ways}$$

Five from A and none from B: This can be done in

$$\binom{7}{5}\binom{5}{0} \qquad \text{ways}$$

So, we have an unexpected identity:

$$\binom{12}{5} = \binom{7}{0}\binom{5}{5}+\binom{7}{1}\binom{5}{4}+\binom{7}{2}\binom{5}{3}+\binom{7}{3}\binom{5}{2}+\binom{7}{4}\binom{5}{1}+\binom{7}{5}\binom{5}{0}$$

Similarly:

$$\binom{12}{5} = \binom{6}{0}\binom{6}{5}+\binom{6}{1}\binom{6}{4}+\binom{6}{2}\binom{6}{3}+\binom{6}{3}\binom{6}{2}+\binom{6}{4}\binom{6}{1}+\binom{6}{5}\binom{6}{0}$$

Once you get in the swing of things, there are many more variations on this theme, so you can invent all kinds of identities (your students will love you for this).

Generalizing these ideas, we have a useful result:

Theorem 2 (Vandermonde's Identity) *If m, n, and r are non-negative integers,*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

Ways to think about it

Combinatorial proofs are wonderful, but they are not the *only* wonderful proofs for identities. There are at least two other methods that deserve mention.

Algebraic proofs: Consider Vandermonde's identity in Theorem 2. One way to prove it is to show that both sides are the coefficient of the same term in an algebraic simplification. Since the coefficient of a term in a polynomial is unique, both sides must be equal. In the Vandermonde case, we start with the fact that

$$(1+x)^{(m+n)} = (1+x)^m(1+x)^n.$$

The idea is to look at the coefficient of x^r on both sides. On the left-hand side, it is, by the binomial theorem, $\binom{m+n}{r}$. On the right-hand side, imagine expanding $(1+x)^m$ and $(1+x)^n$ by the binomial theorem and multiplying the results together. So, the coefficient of x^r on the right-hand side is

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k},$$

and the identity follows.

Computerized proofs: A body of research starting in about the mid-1940s and extending to current work—by the authors of [5] and others—has developed a set of algorithms that can be implemented on a computer and that will supply proofs of binomial coefficient identities. Better yet, if you don't know how to simplify a sum like the right-hand side of Vandermonde's identity, the algorithms will *find* a closed form for you or report with certainty that none exists. The wonderful and readable book [5] contains all the details.

To get an x^r on the right, you'd have to take all products of the coefficients of x^k (from the first expansion) and x^{r-k} (from the second) and add the results. But the coefficient of x^k in $(1+x)^m$ is $\binom{m}{k}$, and the coefficient of x^{r-k} in $(1+x)^n$ is $\binom{n}{r-k}$.

Problems

16. Suppose you have a group of twelve people and you want to form a committee of seven people. The seven people want to pick a subcommittee (from their ranks) of four people. In how many ways can all this be done?

Find at least two ways to count the possibilities. What if you picked the subcommittee first and then "grew" it to the full committee?

17. Show that

$$\binom{12}{7}\binom{7}{4} = \binom{12}{4}\binom{8}{3}$$

18. In general, show that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

In problems 18–20, n, r, and k are integers for which the formulas make sense.

19. Give at least two proofs of the identity

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

Proof #1: It's a special case of problem 18

20. Give at least two proofs of the identity

$$(r+1)\binom{n}{r+1} = (n-r)\binom{n}{r}$$

21. Recall from chapter 1 (section 1.3) the combinatorial polynomials that make up the Mahler basis:

$$\binom{x}{r} = \frac{x(x-1)\dots(x-r+1)}{r!}$$

Give at least two proofs of the identity

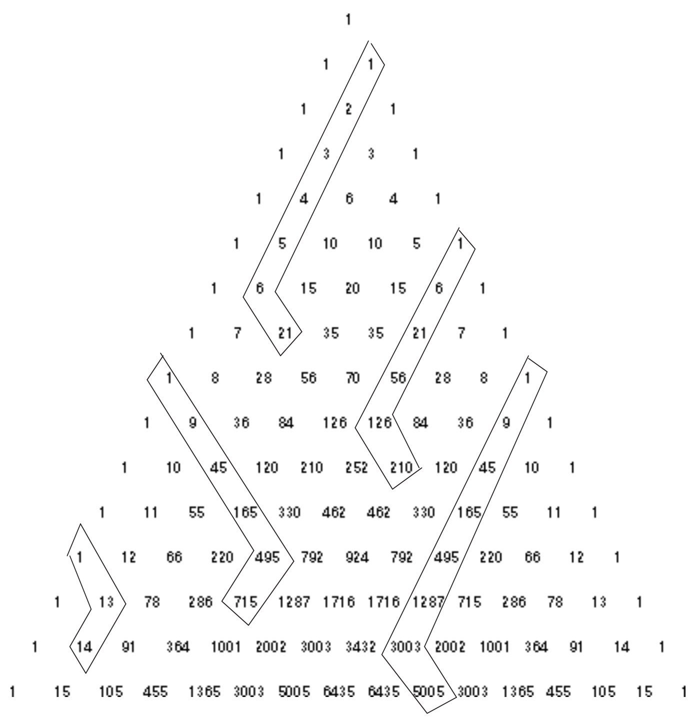
$$\frac{x-r}{r+1}\binom{x}{r} = \binom{x}{r+1}$$

22. Write out the first few powers of 11. Why do their digits give entries in Pascal's triangle? Will they always be entries in Pascal's triangle?

I got this problem from my high school teacher, Frank Kelley.

23. Pascal's triangle enjoys a sort of "hockey stick" property:

The hockey stick property: If you start at the end of any row and draw a hockey stick along a diagonal as shown, the sum of the entries on the handle of the stick is the entry at the tip of the blade.



Express the hockey stick property as an identity involving binomial coefficients and prove the identity.

24. Show that

$$\binom{2n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{n}{r-k}$$

Hint: Split the $2n$-element set into two equal pieces.

25. Show that

$$\binom{3n}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{2n}{r-k}$$

26. Show that

$$\binom{m+n}{r+s} = \sum_{k=-r}^{s} \binom{m}{r+k}\binom{n}{s-k}$$

27. Prove Vandermonde's Identity (Theorem 2).

28. Prove that if n is a positive integer,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

The sum of the squares of the entries in the nth row of Pascal's triangle is the middle entry of the $2n$th row? Wow.

29. Prove the following identity:

$$\frac{\binom{m+1}{k}\binom{k+1}{j}}{\binom{m+1}{j}} = \frac{k+1}{m+2-j}\binom{m+2-j}{k+1-j}$$

30. Show that the number of ways you can roll a total of 5 on two dice is the coefficient of x^5 in

$$(x+x^2+x^3+x^4+x^5+x^6)^2$$

31. What's the number of ways to roll a total of 9 on three dice?

A CAS may help here.

32. What's the sum of *all* the numbers in Pascal's triangle, up to and including the nth row. Prove what you say.

33. Invent three identities involving binomial coefficients by coming up with ways to count the same thing in two different ways.

34. Invent three identities involving binomial coefficients by substituting numbers in the binomial theorem.

35. Give at least two proofs of the fact that

Method #1: Use the binomial theorem.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

36. Suppose A is a set with n elements. Show that the number of subsets of A with an even number of elements is the same as the number of subsets of A with an odd number of elements.

37. Establish the identity

$$\begin{aligned}(x+a+1)^n-(x+a)^n &= nx^{n-1}+\binom{n}{n-2}x^{n-2}\left((a+1)^2-a^2\right)\\ &+\binom{n}{n-3}x^{n-3}\left((a+1)^3-a^3\right)+\cdots+\binom{n}{0}\left((a+1)^n-a^n\right)\end{aligned}$$

Compare with problem 80 on page 123 of Chapter 2.

38. Establish the identity

$$\begin{aligned}&4(x+a)^3+6(x+a)^2+4(x+a)+1=\\ &\quad 4x^3+6x^2\left((a+1)^2-a^2)\right)+4x\left((a+1)^3-a^3)\right)+\left((a+1)^4-a^4)\right)\end{aligned}$$

39. Consider the sequence of numbers defined by

We'll meet this sequence again in the next chapter.

$$B_m = \begin{cases} 1 & \text{if } m = 0 \\ -\dfrac{1}{m+1}\left(\displaystyle\sum_{k=0}^{m-1}\binom{m+1}{k}B_k\right) & \text{if } m > 0 \end{cases}$$

(a) Calculate B_k for $k = 1, \ldots, 6$.
(b) Consider the polynomial in two variables

$$4(x+a)^3 + 6(x+a)^2 + 4(x+a) + 1$$

If this is expanded and each power of a, a^k, is replaced by B_k, show that everything simplifies to $4x^3$.

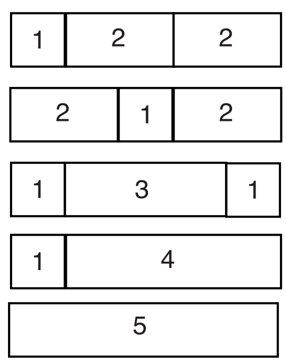
40. Find a formula for the sum of the cardinalities of all the subsets of an n-element set.

The cardinality of a finite set is how many elements it has.

41. Give at least two proofs of the identity

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

42. You can use rods of integer sizes to build "trains" that all share a common length. A "train of length 5" is a row of rods whose combined length is 5. Here are some examples



Notice that the 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately. If you use identical rods in a different order, this is a separate train.
(a) How many trains of length 5 are there?
(b) Come up with a formula for the number of trains of length n. Prove that your formula is correct.
(c) How many trains of length n have k "cars?"

Assume you have rods of every possible integer length available. Can you come up with an *algorithm* that will generate all the trains of length n?

4.2 The Simplex Lock

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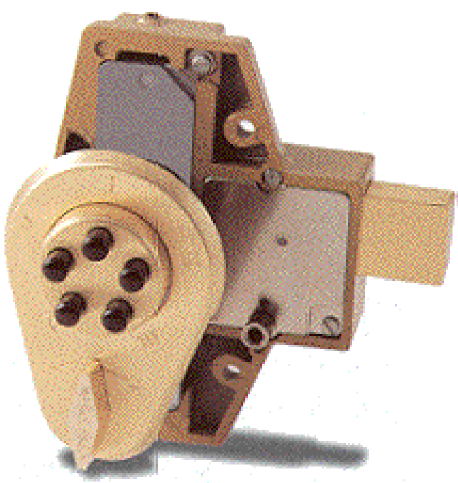
What has come to be known as the *Simplex lock problem* has been a staple in my classes, from elementary algebra to advanced courses, for years. This section takes a somewhat deeper look at the problem, developing some of the (student and teacher) approaches we've seen. More precisely, we'll outline some of these approaches, and leave the fun part (filling in the details) to you.

I got the problem from Brian Harvey, who had already been using it for some time with his computer science students. See problem 49 on page 255 for an example of what one of Brian's students did with the problem

Let's start with the problem. The Simplex company makes a combination lock that is used in many public buildings. It comes in several versions. Here are two:



These 5-button devices are purely mechanical (no electronics). You can set the combination using the following rules:

1. A combination is a sequence of 0 or more pushes, each push involving at least one button.
2. Each button may be used at most once (once you press it, it stays in).
3. Each push may include any number of "open" buttons, from one to five.
4. When two or more buttons are pushed at the same time, order doesn't matter.

Possible combinations:

Notation: $\{\{1,2\},\{3\}\}$ means "press 1 and 2 together, then press 3."

- $\{\{1,2\},\{3\}\}$
- $\{\{2,1\},\{3\}\}$
- $\{\{2\},\{1\},\{3\}\}$
- $\{\{1,2,4\},\{3,5\}\}$
- $\{\{1,2,4,3,5\}\}$
- $\{\{1,2\},\{4\},\{3,5\}\}$
- $\{\{3\},\{1,2\}\}$
- $\{\ \}$
- $\{\ 2\}$

The company advertises thousands of combinations, and (as we say to our students), the question is, "How many combinations are there? Is the company telling the truth?"

Well, they are (barely), but the value of the project lies less in the answer than in the approaches people take to it.

PROBLEM

Stop here, close the book, and don't open it again until you have thought about the problem for at least two days. Reading ahead will spoil the fun and will hamper your creativity.

Almost every time I use this problem with students or teachers, I see an approach that is new to me. It will be fun to compare what you do to the suggestions in the next section.

4.3 Some approaches to the Simplex Lock Problem

Here are some approaches to the Simplex lock problem that I've seen over the years.

Enumeration method number 1. One strategy is to classify the combinations by the total number of buttons used (0, 1, 2, 3, 4, or 5). Then you can sum these answers.

Suppose, for example, your combination uses only two buttons. Then there are only two possible "shapes" the combination could have:

One student used her solution to the problem of determining all the "rod trains" (problem 42 on page 239) of length 5 to determine all the "shapes."

For the first shape, there two pushes. For the first push, there are five possible buttons. After choosing the button used for the first push, there are four buttons left to choose from for the second push.. So, there are $5 \times 4 = 20$ possible combinations for the first kind of combination.

For the second shape, there's only one push, and it contains two buttons. There are five buttons to pick from, so there are

$$\binom{5}{2} = 10$$

possible combinations of this type.

So, you can represent the two button situation this way.

A total of 30 combinations use two buttons.

Where do the numbers come from? For the first shape, you have five buttons and you have to pick one for the first push. There are $\binom{5}{1}$ ways to do this. Then you only have four buttons left, and you have to pick one for the second push. There are $\binom{4}{1}$ ways to do *this*. So, there are

$$\binom{5}{1}\binom{4}{1} = 20$$

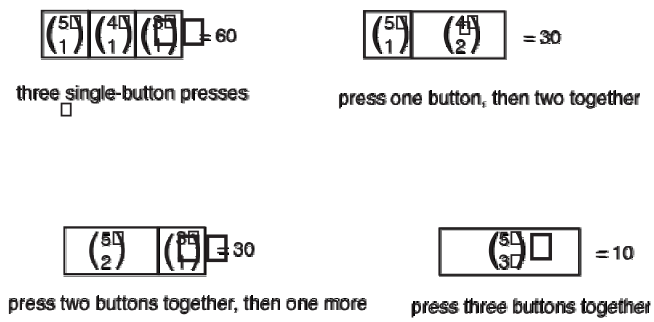
ways to get a combination pressing one button then another.

Using this scheme, the three button situation (which has four possible shapes) would look like this:

Look at the first shape in the second row, for example (press two buttons together, then one more). You have $\binom{5}{2}$ choices for the first push, Then you have to pick one button from the remaining three for the second push. That can be done in $\binom{3}{1}$ ways. So, there are

$$\binom{5}{2}\binom{3}{1} = 30$$

ways to press two buttons and then one more.



A total of 130 presses use three buttons.

The idea is to do this counting scheme for every possible button total from 0 to 5. Here's a summary of what happens:

1. **No buttons**: One combination (the door is always open)
2. **One button**: Five possible combinations ($\binom{5}{1}$).
3. **Two buttons**: We did this above. Two shapes and 30 combinations.
4. **Three buttons**. We did this above. Four shapes and 130 combinations.

We'll leave the remaining two possibilities to you.

Enumeration method number 2. There's another enumeration technique that's a little more abstract. That's good, because abstraction lets you look at a bigger picture. This method will lead to an algorithm that lets you figure the number of combinations on a lock with *any* number of buttons.

The basic idea is to consider locks with different numbers of buttons, and then to count the number of combinations that use *all* the buttons on a given lock. So, we imagine a function L (for lock) so that $L(m)$ is the number of combinations on an m-button lock that use all the buttons. We then try to calculate

$L(1)$,$L(2)$, $L(3)$, $L(4)$, and $L(5)$. How will this help? Well, suppose someone gave you the following table.

These numbers are actually the right ones, as we'll see in problem 45.

n = Number of buttons on the lock	$L(n)$ = number of combinations that use *all* the buttons
0	1
1	1
2	3
3	13
4	75
5	541

Then we could figure out the total number of combinations on a five-button lock by reasoning like this:

- A combination has to use 0, 1, 2, 3, 4, or 5 buttons
- There's one combination that uses no buttons
- To get the number of combinations that use one button, count the number of ways to pick a button (there are $\binom{5}{1} = 5$ ways to do this). On each of these buttons, there is exactly one combination. So, there are $\binom{5}{1} \times 1$ combinations that use exactly one button.
- To get the number of combinations that use two buttons, count the number of ways to pick two buttons to actually use (there are $\binom{5}{2} = 10$ ways to do this). On each of these pairs of buttons, there are exactly three combinations (from the table). So, there are $\binom{5}{2} \times 3 = 30$ combinations that use exactly two buttons.
- To get the number of combinations that use three buttons, count the number of ways to pick three buttons to actually use (there are $\binom{5}{3} = 10$ ways to do this). On each of these triples of buttons, there are exactly 13 combinations (from the table). So, there are $\binom{5}{3} \times 13 = 130$ combinations that use exactly three buttons.
- To get the number of combinations that use four buttons, count the number of ways to pick four buttons to actually use (there are $\binom{5}{4} = 5$ ways to do this). On each of these quadruples of buttons, there are exactly 75 combinations (from the table). So, there are $\binom{5}{4} \times 75 = 375$ combinations that use exactly three buttons.
- To get the number of combinations that use five buttons,

count the number of ways to pick five buttons to actually use (there are $\binom{5}{5} = 1$ way to do this). On each of these quintuples of buttons, there are exactly 541 combinations (from the table). So, there are $\binom{5}{5} \times 541 = 541$ combinations that use exactly three buttons.

- Add these up:

Number of buttons used	number of combinations
0	1
1	$\binom{5}{1} \times 1 = 5$
2	$\binom{5}{2} \times 3 = 30$
3	$\binom{5}{3} \times 13 = 130$
4	$\binom{5}{4} \times 75 = 375$
5	$\binom{5}{5} \times 541 = 541$
total	1082

Whoops—there's the answer: 1082. But, as we said earlier, the thrill is in the chase.

So, the total number of combinations for a 5-button lock is:

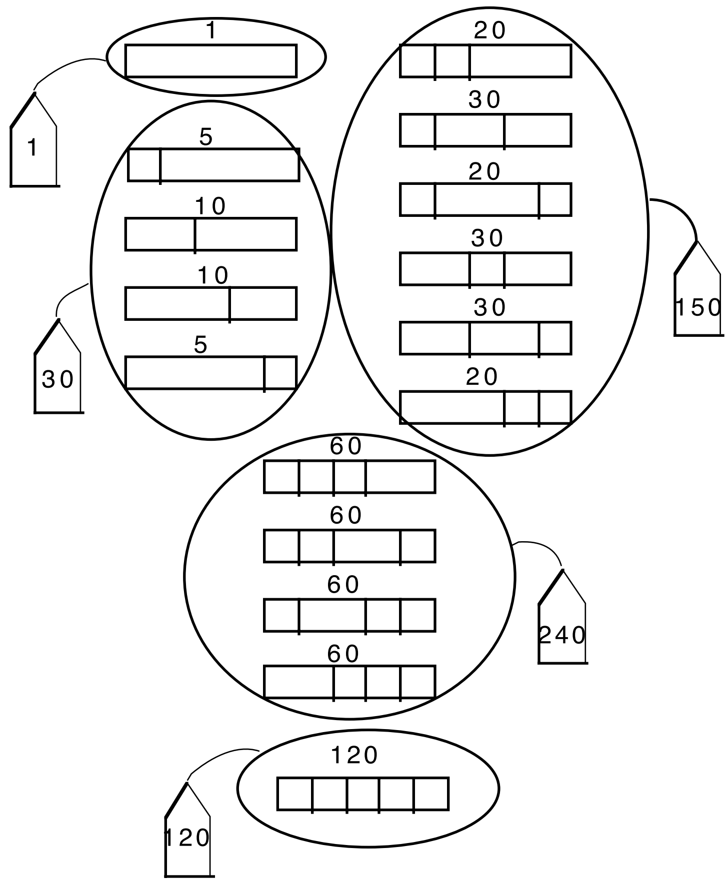
$$\binom{5}{0}L(0)+\binom{5}{1}L(1)+\binom{5}{2}L(2)+\binom{5}{3}L(3)+\binom{5}{4}L(4)+\binom{5}{5}L(5)$$

Ways to think about it

It remains to show that the table on page 244 is correct. That's the object of problem 45. One way to think about this is to count the combinations that use all the buttons on an m-button lock by the number of pushes used. Here's one way to think about it for $m = 5$

The 20 above the first shape in the oval with the 150 tag came from

$$\binom{5}{1}\binom{4}{1}\binom{3}{3}$$



Notice that the number of combinations on an m-button lock that use all the m-buttons and that contain exactly k pushes is none other than the number we denoted by $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ in problem 9 on page 229. Then the discussion above can be summarized in

$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ is the number of *ordered* partitions of an m-element set that have k parts.

a table:

k	$\left\langle \begin{matrix} 5 \\ k \end{matrix} \right\rangle$
0	0
1	1
2	30
3	150
4	240
5	120

and

$$L(5) = \left\langle \begin{matrix} 5 \\ 0 \end{matrix} \right\rangle + \left\langle \begin{matrix} 5 \\ 1 \end{matrix} \right\rangle + \left\langle \begin{matrix} 5 \\ 2 \end{matrix} \right\rangle + \left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle + \left\langle \begin{matrix} 5 \\ 4 \end{matrix} \right\rangle + \left\langle \begin{matrix} 5 \\ 5 \end{matrix} \right\rangle = 541$$

Notice how $L(5)$ is exactly half the total number of combinations on a 5-button lock? Could that be a coincidence?

You could work out this scheme for any number of buttons. We did it for 0–5 and found

Pushes → Buttons ↓	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	2	0	0	0
3	0	1	6	6	0	0
4	0	1	14	36	24	0
5	0	1	30	150	240	120

These are lovely numbers,and the table contains many patterns that may or may not hold in general.

Ways to think about it

The table above is (ignoring the 0s) an "$\left\langle {m \atop k} \right\rangle$ triangle." One way to think about generating the entries in the $\left\langle {m \atop k} \right\rangle$ triangle is to mimic what was done in the diagram on page 246 for five buttons. Another is to think about how you'd build a row from the previous row. For example, knowing the fifth row, how could you calculate, say, $\left\langle {6 \atop 4} \right\rangle$. Well, $\left\langle {6 \atop 4} \right\rangle$ counts the number of combinations on a 6-button lock that uses four pushes. Suppose the buttons are numbered 1–6, and concentrate on button number 6. Button 6 could show up in a push with other buttons or as a push all by itself.

- Suppose it is in a push with other buttons. Then if you "forget the 6," you have a combination on 1–5 with 4 pushes.
- Suppose it is a push all by itself. Then if you "forget the 6," you have a combination on 1–5 with 3 pushes.

Now imagine working from five buttons to six. You could get a 4-push 6-button combination from looking at the combinations that use 1–5 and adding in the number 6 button, either in an existing push or as a push all by itself.

- There are

$$\left\langle {5 \atop 4} \right\rangle$$

combinations on 1–5 that use four pushes, and you could put the 6 in any one of those pushes. Hence there are $4\left\langle {5 \atop 4} \right\rangle$ =960 combinations on six buttons for which 6 is in an existing push.

And. . .

This method generalizes to an approach to problem 47 on page 254.

- There are $\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle$ combinations on 1–5 that use three pushes, and you could put 6 in its own the push at the start of the combination, between any two pushes, or at the end of the combination. You can check that for a combination with three pushes, there are four "holes" in which you can put an extra push—at the front, between the first two pushes, between the second two pushes, or at the end. Hence there are $4\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle = 600$ combinations on six buttons for which 6 is in an existing push.

The punchline is that

$$\left\langle \begin{matrix} 6 \\ 4 \end{matrix} \right\rangle = 4\left\langle \begin{matrix} 5 \\ 4 \end{matrix} \right\rangle + 4\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle = 1560$$

Combinatorics. People who have worked with the previous method and similar ones have abstracted some patterns in the calculations that have allowed them to deal with the problem in more generality (any number of buttons) and at a more abstract level (without all the arithmetic with specific numbers and all the calculations with the various shapes).

Giving names to things often helps with the abstraction: you name something, then, making believe it's a real thing, you look for some of its properties. For example:

Let $T(m)$ = the number of combinations on an m-button lock, and remember that
$L(m)$ = the number of combinations on an m-button lock that use all m buttons.

Think of $T(m)$ and $L(m)$ as unknowns that you (desperately) want to find. Since there's two of them, you'll need two equations relating them. If you've mucked around with special cases, you may have noticed this:

Theorem 3

$$T(m) = \sum_{k=0}^{m} \binom{m}{k} L(k).$$

Proof. For each k between 0 and m, here's a two-step way to count all the combinations that use exactly k buttons.:

A combinatorial proof again.

1. Pick k buttons to actually use in the combination. This can be done in $\binom{m}{k}$ ways.
2. Once you pick the buttons to use, count all the combinations that use all k buttons you have picked. This can be done in $L(k)$ ways.

So, there are $\binom{m}{0}L(0)$ combinations that use no buttons, $\binom{m}{1}L(1)$ that use one button, $\binom{m}{2}L(2)$ that use two buttons, and so on. Add them up and you get the total number of combinations that use *any* number of buttons. That's what the theorem claims:

$$T(m) = \sum_{k=0}^{m} \binom{m}{k} L(k). \qquad \textbf{Q.E.D.}$$

That's one equation. For the second, notice that for a five-button lock, the number of combinations that use all five buttons is 541, exactly half the total number of combinations. If you play with other numbers of buttons, this seems to be the case in general. In other words

Theorem 4

$$T(m) = 2L(m).$$

Proof. A common way to show that two sets have the same size is to set up a one-to-one correspondence between them. That is, If you have one set, P, and another set, R, and you want to show that they have the same number of elements, set up a function that sends the elements of P to the elements of R so that every element of P is assigned to exactly one element of R and all the elements of R are hit.

Another way to think of this result is that the number of combinations that use all the buttons is the same as the number of combinations that use fewer than all the buttons.

Imagine P is the set of combinations that use all m buttons, and let R be the set of combinations that use fewer than m buttons. For each combination in P, you can get a combination in R by "forgetting the last push." For example, if $m = 5$ and you have

$$\{1,2,4\}\{5\}\{3\}$$

you assign it to

$$\{1,2,4\}\{5\}.$$

Now,

- This assignment will produce every combination that uses fewer than m buttons. Why? Well, take a combination that uses fewer than m buttons. Take all the buttons that it doesn't use and make a new push from those, putting it on the end of what you have. This produces a combination that uses all the buttons and that gets assigned to the one you started with.
- No combination that uses fewer than m buttons gets "hit" twice by this assignment. Why? Well, two combinations that use all m buttons that look the same when you forget the last push must differ only in the last push. But you have no *choice* for the last push of a combination that uses all m buttons: it has to contain all the buttons that aren't in the previous pushes.

It follows (if you think about it) that our assignment is one-to-one, and the theorem is proved. **Q.E.D.**

Well, we have two equations:

$$\begin{aligned} T(m) &= 2L(m) \quad \text{and} \\ T(m) &= \textstyle\sum_{k=0}^{m} \binom{m}{k} L(k) = \binom{m}{0} L(0) + \binom{m}{1} L(1) + \cdots + \binom{m}{n-1} L(m-1) + \binom{m}{m} L(m). \end{aligned}$$

So,

$$2L(m) = \binom{m}{0} L(0) + \binom{m}{1} L(1) + \cdots + \binom{m}{m-1} L(m-1) + \binom{m}{m} L(m).$$

The last term on the right is just $L(m)$; subtract it from both sides and get

$$L(m) = \binom{m}{0} L(0) + \binom{m}{1} L(1) + \cdots + \binom{m}{m-1} L(m-1).$$

What a beautiful recurrence. It will let us calculate the $L(m)$ in terms of previous values of $L(k)$. And you can double each $L(m)$ to get the total number of combinations $T(m)$ on an m-button lock. Look:

$$
\begin{aligned}
L(0) &= 1 \\
L(1) &= \tbinom{1}{0}L(0) = 1; \quad T(1) = 2 \\
L(2) &= \tbinom{2}{0}L(0) + \tbinom{2}{1}L(1) = 3; \quad T(2) = 6 \\
L(3) &= \tbinom{3}{0}L(0) + \tbinom{3}{1}L(1) + \tbinom{3}{2}L(2) = 13; \quad T(3) = 26 \\
L(4) &= \tbinom{4}{0}L(0) + \tbinom{4}{1}L(1) + \tbinom{4}{2}L(2) + \tbinom{4}{3}L(3) = 75; \quad T(4) = 150 \\
L(5) &= \tbinom{5}{0}L(0) + \tbinom{5}{1}L(1) + \tbinom{5}{2}L(2) + \tbinom{5}{3}L(3) + \tbinom{5}{4}L(4) = 541; \quad T(5) = 1082
\end{aligned}
$$

Let's state all this as a theorem:

Theorem 5

The number of combinations on an m-button lock ($m > 0$) is $2L(m)$ where

$$
\begin{aligned}
L(0) &= 1 \quad \textit{and} \\
L(m) &= \sum_{k=0}^{m-1} \binom{m}{k} L(k) \quad \textit{if } m > 0.
\end{aligned}
$$

Theorem 5 allows you to quickly calculate the values of L (and hence, of T). A CAS or calculator makes the job even easier. For your enjoyment, here are the first 16 values:

m	$L(m)$	$T(m)$
0	1	1
1	1	2
2	3	6
3	13	26
4	75	150
5	541	1082
6	4683	9366
7	47293	94586
8	545835	1091670
9	7087261	14174522
10	102247563	204495126
11	1622632573	3245265146
12	28091567595	56183135190
13	526858348381	1053716696762
14	10641342970443	21282685940886
15	230283190977853	460566381955706

Notice that the equation $T(m) = 2L(m)$ doesn't hold if $m = 0$. Where does the proof of Theorem 4 break down if $m = 0$? The case $m = 0$ is sometimes called the "pathological case." Why?

Approaches to the Simplex lock problem abound. We suggest a couple more in the following problems. See [3] for an interesting paper on the problem.

Problems

43. Find a method not described in this section for counting the number of combinations on a Simplex lock.

44. Finish the details of enumeration method 1 (page 242).

45. Show that the numbers in the table on page 244 are correct.

46. Show that the numbers in the table on page 247 are correct, and extend the table for three more rows.

47. Prove Theorem 6

That is, $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ is the number of combinations on an m-button lock that use all the m-buttons and that contain exactly k pushes.

Theorem 6

The numbers $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ *defined on page 246 satisfy the recurrence*

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = \begin{cases} 1 & \text{if } m = k = 0, \\ 0 & \text{if } k > m \text{ or } k < 0, \\ k\left(\left\langle \begin{matrix} m-1 \\ k-1 \end{matrix} \right\rangle + \left\langle \begin{matrix} m-1 \\ k \end{matrix} \right\rangle\right) & \text{if } 0 \le k \le m. \end{cases}$$

48. Suppose we define a function γ of two variables, built by a "convolution" of lock numbers and binomial coefficients:

$$\begin{aligned} \gamma(n,m) &= \sum_{k=1}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n}{k} \\ &= \left\langle \begin{matrix} m \\ 1 \end{matrix} \right\rangle \binom{n}{1} + \left\langle \begin{matrix} m \\ 2 \end{matrix} \right\rangle \binom{n}{2} + \cdots + \left\langle \begin{matrix} m \\ m \end{matrix} \right\rangle \binom{n}{m} \end{aligned}$$

Investigate the function γ, perhaps by filling in this table:

$m \rightarrow$ $n \downarrow$ $\gamma(n,m)\searrow$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Explain anything that you see.

49. Brian Harvey, who teaches computer science at Berkeley and who first put me onto the Simplex lock problem in 1986, often teaches summer courses to high school students from the area. He reports on the work of one student from Oakland:

Brian is the author of a 3-volume set of books [4] that teach serious programming—and a ton of mathematics—primarily to high school students.

> "One of my high school students just came in with the following idea: You make a picture like Pascal's triangle, except that in adding entries from one row to the next, each number on row N has a weight, which is its position within the row (starting from 1). So

Notice that the sum of the entries in row n is $T(n)$, not $L(n)$. Brian's student arrived at his method by looking for patterns. The justification came later.

$$
\begin{array}{ccccccccc}
 & & & & 1_{[1]} & & & & \\
 & & & 1_{[1]} & & 1_{[2]} & & & \\
 & & 1_{[1]} & & 3_{[2]} & & 2_{[3]} & & \\
 & 1_{[1]} & & 7_{[2]} & & 12_{[3]} & & 6_{[4]} & \\
1_{[1]} & & 15_{[2]} & & 50_{[3]} & & 60_{[4]} & & 24_{[5]}
\end{array}
$$

> etc. The numbers in brackets are the weights. So for example the 7 in the fourth row is $1 \cdot 1 + 3 \cdot 2$.

> The 50 in the fifth row is $7 \cdot 2 + 12 \cdot 3$. And the sum of each row is (starting at $n = 0$) the number of combinations in an n-button Simplex lock."

Investigate this method. If $P(n, k)$ stands for the element in the nth row and kth column (Pascal's triangle style), express the student's recurrence in symbols. Does the method work? Do the numbers in each row have any "lock-theoretic" significance?

50. Here's an unusual method for generating $L(n)$. See if you can come up with an argument for why it works: Consider the operation Λ on polynomials where

$$\Lambda(f)(x) = (x + 1)\, f(x + 1) + x\, f(x)$$

Λ is a creature similar to Δ:

$$\begin{aligned} \Delta(f)(x) &= f(x+1) - f(x) \text{ and} \\ \Lambda(f)(x) &= (x+1)\, f(x+1) + x\, f(x) \end{aligned}$$

So, for example,

$$\begin{aligned} \Lambda(2x + 1) &= (x + 1)\,(2(x + 1) + 1) + x(2x + 1) \\ &= 4x^2 + 6x + 3 \end{aligned}$$

Next, Consider the sequence of polynomials $s(n)$ generated by the following rule:

Why would anyone ever think up an operation like Λ? Or the polynomials $s(n)$?

$$s(n) = \begin{cases} 1 & \text{if } n = 0 \\ \Lambda(s(n-1)) & \text{if } n > 0 \end{cases}$$

A CAS quickly generates some of the $s(n)$:

n	$s(n)$
0	1
1	$1 + 2x$
2	$3 + 6x + 4x^2$
3	$13 + 30x + 24x^2 + 8x^3$
4	$75 + 190x + 180x^2 + 80x^3 + 16x^4$
5	$541 + 1470x + 1560x^2 + 840x^3 + 240x^4 + 32x^5$
6	$4683 + 13454x + 15540x^2 + 9520x^3 + 3360x^4 + 672x^5 + 64x^6$

Well, look at that: the constant term of $s(n)$ seems to be $L(n)$—the number of combinations on an n-button lock that use all the buttons. Is it true? If so, why? If not, when does it break down?

Find and explain some interesting patterns in the sequence of polynomials. What happens if you evaluate each polynomial at 1? To see what's going on, it really helps to generate a few by hand.

4.4 Connections to the Mahler Basis

This section was motivated by a seeming coincidence. In Chapters 1 and 2, we investigated ways to represent polynomials in different "bases." The usual way to express a polynomial is as a linear combination of powers of x. For example, suppose

$$f(x) = \frac{x^5}{40} - \frac{x^4}{3} + \frac{53x^3}{24} - \frac{14x^2}{3} - \frac{37x}{30} + 7.$$

For some purposes (Newton's difference formula, for example), it turns out to be more useful to express f in terms of the Mahler basis

This is problem 66 on page 60 of Chapter 1.

$$f(x) = 3\binom{x}{5} - 2\binom{x}{4} + 5\binom{x}{3} - 4\binom{x}{1} + 7\binom{x}{0}.$$

These Mahler polynomials take values of binomial coefficients at non-negative integers. When expanded, they look like this:

k	$\binom{x}{k}$	Expanded version
0	1	1
1	x	x
2	$\frac{x(x-1)}{2!}$	$\frac{-x+x^2}{2}$
3	$\frac{x(x-1)(x-2)}{3!}$	$\frac{2\,x-3\,x^2+x^3}{6}$
4	$\frac{x(x-1)(x-2)(x-3)}{4!}$	$\frac{-6\,x+11\,x^2-6\,x^3+x^4}{24}$
5	$\frac{x(x-1)(x-2)(x-3)(x-4)}{5!}$	$\frac{24\,x-50\,x^2+35\,x^3-10\,x^4+x^5}{120}$
6	$\frac{x(x-1)(x-2)(x-3)(x-4)(x-5)}{6!}$	$\frac{-120\,x+274\,x^2-225\,x^3+85\,x^4-15\,x^5+x^6}{720}$

The problems in section 1.6 of Chapter 1 looked at the conversion formulas between "normal" and Mahler bases. In particular, problem 61 showed that

Problem 63 produced a recursive method for generating the $a_{m,k}$.

$$x^m = \sum_{k=0}^{m} a_{m,k} \binom{x}{k}$$

where

. . . and 0^0 is taken to be 1.

$$a_{m,k} = k^m - \binom{k}{1}(k-1)^m + \binom{k}{2}(k-2)^m - \cdots + (-1)^{k-1}\binom{k}{k-1}1^m.$$

We can calculate the coefficients $a_{m,k}$ and put them in a matrix (here, up to x^5):

$\binom{x}{k} \rightarrow$ $x^m \downarrow \quad a_{m,k} \searrow$	$\binom{x}{0}$	$\binom{x}{1}$	$\binom{x}{2}$	$\binom{x}{3}$	$\binom{x}{4}$	$\binom{x}{5}$
x^0	1	0	0	0	0	0
x^1	0	1	0	0	0	0
x^2	0	1	2	0	0	0
x^3	0	1	6	6	0	0
x^4	0	1	14	36	24	0
x^5	0	1	30	150	240	120

So, for example,

To check this, write each $\binom{x}{k}$ as $\frac{x(x-1)\ldots(x-k+1)}{k!}$ and simplify.

$$x^4 = 0\binom{x}{0} + 1\binom{x}{1} + 14\binom{x}{2} + 36\binom{x}{3} + 24\binom{x}{4}.$$

Oh my. Look back at the table on page 247:

Pushes (k) → Buttons (m) ↓ $\left\langle {m \atop k} \right\rangle \searrow$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	2	0	0	0
3	0	1	6	6	0	0
4	0	1	14	36	24	0
5	0	1	30	150	240	120

This is a table of the $\left\langle {m \atop k} \right\rangle$, the number of combinations on an m-button lock that use all m buttons and have k-pushes. But it seems to be the same (well, it *is* the same up to 5) as the table for the $a_{m,k}$, the conversion coefficients that express powers of x as a Mahler expansion!

Why should this be so? One reason is buried in the problem sets. Problem 63 on page 59 of Chapter 1 showed that the $a_{m,k}$ are determined by the relations:

$$a_{m,k} = \begin{cases} 1 & \text{if } m = k = 0, \\ 0 & \text{if } k > m \text{ or } k < 0, \\ k\left(a_{m-1,k-1} + a_{m-1,k}\right) & \text{if } 0 \le k \le m. \end{cases}$$

And by Theorem 6 on page 254 of this chapter, you showed that

Maybe it $isn't$ always the same—mathematics is full of examples of tables that match for a long time but then disagree. We even saw how to produce such false matches in Chapter 1. One needs to be careful not to jump to conclusions. But, in this case, we have a genuine theorem, and a surprising one at that. Angelo DiDomenico, a retired (after over 40 years) high school teacher in Massachusetts says, "Mathematics is generous. It gives you more than you ask for."

the numbers $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ satisfy the recurrence

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = \begin{cases} 1 & \text{if } m = k = 0, \\ 0 & \text{if } k > m \text{ or } k < 0, \\ k\left(\left\langle \begin{matrix} m-1 \\ k-1 \end{matrix} \right\rangle + \left\langle \begin{matrix} m-1 \\ k \end{matrix} \right\rangle\right) & \text{if } 0 \le k \le m. \end{cases}$$

Since the two functions are defined by the same recursive definition, they are equal for all pairs (m, k) in their domain. The details of the proof are fussy, but essentially, it's a proof by induction on m (see problem 51).

Let's state our result in a slightly different way as a theorem:

Theorem 7

If m, k are non-negative integers, then we have a polynomial identity

$$x^m = \sum_{k=0}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{x}{k}.$$

This is just another way to say that $a_{m,k} = \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$.

So, the coefficient of $\binom{x}{k}$ needed to express x^m in terms of the Mahler basis is just the number of combinations on an m-button Simplex lock that use all the buttons and k pushes.

Proof. We've already outlined one proof that uses induction (problem 51). Let's look at a more combinatorial proof. We start with something you may have established in problem 48 on page 254:

Lemma 1

For any positive integers n, m,

$$n^m = \sum_{k=0}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n}{k}.$$

Proof: By problem 11 on page 229, the left-hand side counts the number of functions from an m element set (say T) to an n-element set (say, S).

But to build such a function, you have to pick some subset of elements in S to actually get hit. Say you pick k elements (there are $\binom{n}{k}$ ways to do this). Once you pick a "target" k-element subset of S, you have to pick some elements of T to go to the first element of the target, some more to go to the second element of S, …, and the remaining elements go to the kth element of the target. This is the same as building a k-push combination— you have to pick some elements of T to go into the first push, some more to go into the second push, …, and the remaining elements go into the kth push. So, there are $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ ways to build the function. Hence there are

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n}{k}$$

functions that have a k-element target. And therefore there are

$$\sum_{k=0}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{n}{k}$$

functions in all. **Q.E.D.**

Now for the proof of Theorem 7, note that the polynomials x^m and

$$\sum_{k=0}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{x}{k}$$

agree for every non-negative integer n, so, they are equal as polynomials by the "function implies form theorem" (Corollary 3 on page 100 of Chapter 2).

Corollary 2

For non-negative m and k,

$$a_{m,k} = \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle .$$

From now on, we'll use both notations—$a_{m,k}$ and $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$—interchangeably.

Proof. The polynomials $\sum_{k=1}^{m} a_{m,k}\binom{x}{k}$ and

$$\sum_{k=0}^{m} \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle \binom{x}{k}$$

are both equal to x^m, so they are equal. This implies their "coefficients" are equal (Chapter 1, problem 69 on page 60). **Q.E.D.**

Finally, using problem 61 on page 59 of Chapter 1, we have a new way to calculate the lock numbers:

Corollary 3

For non-negative m and k,

We need to take 0^0 to be 1 here.

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = k^m - \binom{k}{1}(k-1)^m + \binom{k}{2}(k-2)^m - \cdots + (-1)^{k-1}\binom{k}{k-1}1^m.$$

Ways to think about it

So, we have found two explanations for the fact that the same numbers show up in what seem to be completely different contexts. Have we "explained" the surprise? From a mathematical point of view, of course we have. The combinatorial proof, especially, seems to shed light on the mystery: the number of combinations on an m-button lock that use all m buttons and that have k pushes is just the number of functions from an m-element set to a k-element that "hit" each of the k elements at least once.

But, even after you've been through the proofs and have explained away the mystery, doesn't it still seem eerie that these numbers show up in such different situations?

Problems

51. Use the facts that

$$a_{m,k} = \begin{cases} 1 & \text{if } m = k = 0 \\ 0 & \text{if } k > m \text{ or } k < 0 \\ k\left(a_{m-1,k-1} + a_{m-1,k}\right) & \text{if } 0 \le k \le m \end{cases}$$

and

$$\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle = \begin{cases} 1 & \text{if } m = k = 0 \\ 0 & \text{if } k > m \text{ or } k < 0 \\ k\left(\left\langle \begin{matrix} m-1 \\ k-1 \end{matrix} \right\rangle + \left\langle \begin{matrix} m-1 \\ k \end{matrix} \right\rangle\right) & \text{if } 0 \le k \le m \end{cases}$$

to give a proof by induction (on m) that $a_{m,k} = \left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$ for all non-negative integers m and k.

52. Show that the number of functions from an m element set *onto* a k-element set is $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$.

Problem 15 on page 230 again. A function is "onto" if it "uses up" its range—every element in the range is the image of something in the domain.

53. Show that the number of functions from an m element set to a k-element set that are *not* onto is:

$$\binom{k}{1}(k-1)^m - \binom{k}{2}(k-2)^m + \cdots + (-1)^k\binom{k}{k-1}1^m$$

In Chapter 1, you looked at the problem of "going the other way:" converting the Mahler polynomials back to powers of x. In problems 62 and 64 on page 59 of Chapter 1, you showed that if

If you didn't look at these problems at the time, it would be a good thing to do now.

$$\begin{aligned} \binom{x}{m} &= \frac{x(x-1)(x-2)(x-3)\ldots(x-m+1)}{m!} \\ &= \sum_{k=0}^{m} b_{m,k}x^k, \end{aligned}$$

then the $b_{m,k}$ satisfy the recurrence:

$$b_{m,k} = \begin{cases} 1 & \text{if } m = k = 0 \\ 0 & \text{if } k > m \text{ or } k < 0 \\ \dfrac{b_{m-1,k-1} - (m-1)b_{m-1,k}}{m} & \text{if } 0 \le k \le m \end{cases}$$

54. Fill in the next four rows of the $b_{m,k}$ table. Describe and explain any patterns you see:

Concentrate on the repetition—the rhythm of the calculations.

$x^k \rightarrow$ $\binom{x}{m} \downarrow$ $b_{m,k} \searrow$	x^0	x^1	x^2	x^3	x^4	x^5
$\binom{x}{0} = 1$	1	0	0	0	0	0
$\binom{x}{1} = x$	0	1	0	0	0	0
$\binom{x}{2} = \frac{x(x-1)}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$\binom{x}{3} = \frac{x(x-1)(x-2)}{6}$	0	$\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{6}$	0	0
$\binom{x}{4} = \frac{x(x-1)(x-2)(x-3)}{24}$	0	$-\frac{1}{4}$	$\frac{11}{24}$	$-\frac{1}{4}$	$\frac{1}{24}$	0
$\binom{x}{5} = \frac{x(x-1)(x-2)(x-3)(x-5)}{120}$	0	$\frac{1}{5}$	$-\frac{5}{12}$	$\frac{7}{24}$	$-\frac{1}{12}$	$\frac{1}{120}$

Problems 55–58 **look at the relationship between the** $a_{m,k}$ **and the** $b_{m,k}$**.**

55. Simplify

A problem with a point...

$$a_{3,0}\left(b_{0,0}\right) + a_{3,1}\left(b_{1,0} + b_{1,1}x\right) + a_{3,2}\left(b_{2,0} + b_{2,1}x + b_{2,2}x^2\right)$$
$$+ a_{3,3}\left(b_{3,0} + b_{3,1}x + b_{3,2}x^2 + b_{3,3}x^3\right)$$

56. Simplify

$$a_{m,0}\,(b_{0,0}) + a_{m,1}\,(b_{1,0} + b_{1,1}x) + a_{m,2}\,(b_{2,0} + b_{2,1}x + b_{2,2}x^2) + \cdots + a_{m,m}\,(b_{m,0} + b_{m,1}x + \cdots + b_{m,m}x^m)$$

57. Simplify

$$b_{m,0}\left(a_{0,0}\binom{x}{0}\right) + b_{m,1}\left(a_{1,0}\binom{x}{0} + a_{1,1}\binom{x}{1}\right)$$
$$+b_{m,2}\left(a_{2,0}\binom{x}{0} + a_{2,1}\binom{x}{1} + a_{2,2}\binom{x}{2}\right) + \cdots$$
$$+b_{m,m}\left(a_{m,0}\binom{x}{0} + a_{m,1}\binom{x}{1} + \cdots + a_{m,m}\binom{x}{m}\right)$$

We assume in problem 58 that you know how to multiply matrices. If that's not familiar, all you need is the very basics, and that's something you can look up in any linear algebra book (and in many high school algebra books).

58. Calculate the matrix products

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} & 0 \\ 0 & \frac{1}{5} & -\frac{5}{12} & \frac{7}{24} & -\frac{1}{12} & \frac{1}{120} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 6 & 6 & 0 & 0 \\ 0 & 1 & 14 & 36 & 24 & 0 \\ 0 & 1 & 30 & 150 & 240 & 120 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 6 & 6 & 0 & 0 \\ 0 & 1 & 14 & 36 & 24 & 0 \\ 0 & 1 & 30 & 150 & 240 & 120 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} & 0 \\ 0 & \frac{1}{5} & -\frac{5}{12} & \frac{7}{24} & -\frac{1}{12} & \frac{1}{120} \end{pmatrix}$$

Will the same phenomenon hold for larger sizes?

The numbers $a_{m,k}$ (or $\left\langle \begin{matrix} m \\ k \end{matrix} \right\rangle$, depending on your preference) and $b_{m,k}$ are related to some famous numbers in combinatorics known as the *Stirling numbers*. There are two kinds of Stirling numbers, cleverly called Stirling numbers of the first and second

See the beautiful book [3] for a detailed account of Stirling numbers and many more topics that are connected to this chapter.

kinds. There are many ways to define them, but here's one that's related to the topic of this section:

Recall from Chapter 2, section 2.3, page 109, the notion of "falling powers:"

So, $x^{\underline{m}} = m!\binom{x}{m}$.

$$x^{\underline{m}} = x(x-1)(x-2)(x-3)\cdots(x-m+1)$$

One way to think about Stirling numbers is as conversion factors between powers of x and falling powers of x. More precisely:

1. The Stirling numbers of the first kind, $\left|\begin{matrix} m \\ k \end{matrix}\right|$, are defined by the equation

$$x^{\underline{m}} = \sum_{k=0}^{m} (-1)^{m-k} \left|\begin{matrix} m \\ k \end{matrix}\right| x^k$$

The alternating sign is separated out to keep the Stirling numbers positive, because they have other combinatorial uses. See problem 66 for an example.

2. The Stirling numbers of the second kind, $\left\{\begin{matrix} m \\ k \end{matrix}\right\}$, are defined by the equation

$$x^m = \sum_{k=0}^{m} \left\{\begin{matrix} m \\ k \end{matrix}\right\} x^{\underline{k}}$$

59. Generate a few rows of the table for $\left|\begin{matrix} m \\ k \end{matrix}\right|$. Find and explain some patterns in the table.

60. Show that

$$(-1)^{m+k} m!\, b_{m,k} = \left|\begin{matrix} m \\ k \end{matrix}\right|$$

61. Show that the Stirling numbers of the first kind satisfy the recurrence

$$\left|\begin{matrix} m \\ k \end{matrix}\right| = (m-1)\left|\begin{matrix} m-1 \\ k \end{matrix}\right| + \left|\begin{matrix} m-1 \\ k-1 \end{matrix}\right|$$

62. Generate a few rows of the table for $\left\{\begin{matrix} m \\ k \end{matrix}\right\}$. Find and explain some patterns in the table.

63. Show that

$$\left\langle\begin{matrix} m \\ k \end{matrix}\right\rangle = k!\left\{\begin{matrix} m \\ k \end{matrix}\right\}$$

64. Show that the Stirling numbers of the second kind satisfy the recurrence

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} m-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} m-1 \\ k-1 \end{matrix} \right\}$$

65. Show that

$$\left\{ \begin{matrix} m \\ k \end{matrix} \right\} = c_{m,k}$$

where $c_{m,k}$ is defined in problem 8 on page 228.

See problem 66 if you're stuck.

66. Imagine a variant on the Simplex lock where the order of the pushes doesn't matter.

(a) How many combinations are there on a five-button lock of this type that use all five buttons and have three pushes.

(b) How many combinations are there on a five-button lock of this type that use all five buttons?

(c) How many combinations are there on a five-button lock of this type?

67. Imagine a variant on the Simplex lock where the order of the pushes doesn't matter, but the order that you press buttons within a push *does* matter.

(a) How many combinations are there on a five-button lock of this type that use all five buttons and have three pushes.

(b) How many combinations are there on a five-button lock of this type that use all five buttons?

(c) How many combinations are there on a five-button lock of this type?

So, $[1,4,2][3,5]$ is the same as $[3,5][1,4,2]$ but *different* from $[1,2,4][3,5]$. There'd have to be some sort of signal to show that a push is over, and the lock would just store the collection of pushes to see if it matched the combination.

Notes for Selected Problems

Notes for problem 6 **on page 228.** The definition that we will use is

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Thus, we have

$$\binom{-5}{3} = \frac{(-5)(-6)(-7)}{3!} = -35$$

We can also "back-track" up the rows of Pascal's Triangle, using the fact that any interior number is the sum of the two above it. Below is the regular Pascal's Triangle "left justified" and zeros to the right of the right-most 1.

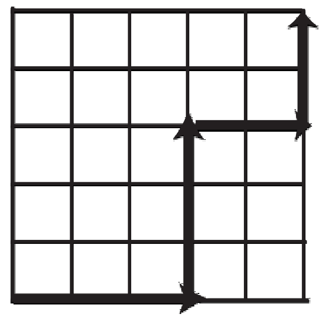
1	0	0	0	0
1	1	0	0	0
1	2	1	0	0
1	3	3	1	0
1	4	6	4	1

Now for the (−1)st row. It must begin with a 1, and the sum of any two numbers equals the number below them.

Row −1	1	-1	1	-1	1	. . .
Row 0	1	0	0	0	0	. . .
Row 1	1	0	0	0	0	. . .
Row 2	1	2	1	0	0	. . .
Row 3	1	3	3	1	0	. . .
Row 4	1	4	6	4	1	. . .

Try expanding the triangle in this fashion and you should see that the third number on the (−5)th row equals −35.

Notes for problem 10 **on page 229.** Any route that Ms. D'Amato takes involves five east moves and five north moves. For example, the route below can be represented by "EEENNNEENN."



Thus, we need to find the number of 10-letter words using five E's and five N's. This is given by

$$\frac{10!}{5!5!} = 252$$

Notes for problem 11 **on page 229.** Let T be an m-element set and S be an n-element set. Thus,

$$T = \{a_1, a_2, \ldots, a_m\} \quad \text{and} \quad S = \{b_1, b_2, \ldots, b_n\}$$

Suppose f is a function from T to S. There are n possibilities for $f(a_1)$, namely $f(a_1) = b_1$, $f(a_1) = b_2$, $\ldots$, or $f(a_1) = b_n$. Likewise, there are n possibilities for $f(a_2)$, n possibilities for $f(a_3)$, and so on. Therefore, there are

$$\underbrace{n \times n \times \cdots \times n}_{m \text{ times}} = n^m$$

possible functions from T to S.

Notes for problem 20 **on page 236.** Suppose you have a group of twelve people and you want to form a committee of eight people. The eight people want to pick a leader of their committee. This can be done in

$$8 \cdot \binom{12}{8} \quad \text{ways.}$$

$\binom{12}{8}$ represents the number of 8-person committees. Multiply it by 8 because any one of the eight members of a committee can be the leader.

We can also pick the seven non-leaders of the committee and choose a leader amongst the five remaining people. This can be done in

$$5 \cdot \binom{12}{7} \quad \text{ways.}$$

Therefore, we have the identity

$$8 \cdot \binom{12}{8} = 5 \cdot \binom{12}{7}$$

It's not hard to see that this can be generalized to the desired formula.

Notes for problem 23 **on page 237.** The hockey stick property can be expressed by

$$\sum_{k=j}^{n-1} \binom{k}{j} = \binom{n}{j+1}$$

where j is a non-negative integer less than n. This identity can be proven by induction on n and by using the formula

$$\binom{n}{j+1} = \binom{n-1}{j} + \binom{n-1}{j+1}$$

Notes for problem 29 **on page 238.** First, we must assume that $k+1-j \geq 0$. Otherwise, $\binom{m+2-j}{k+1-j}$ will have no meaning.

We'll leave it up to you to prove the identity for the special cases $k+1-j=0$ (or $k+1=j$) and $k+1-j=1$ (or $k=j$).

So, now assume that $k+1-j>1$ or $k>j$. Note that

$$\begin{aligned}
\binom{m+1}{k} &= \frac{(m+1)(m)(m-1)\cdots(m+2-k)}{k(k-1)(k-2)\cdots 1} \\
\binom{k+1}{j} &= \frac{(k+1)(k)(k-1)\cdots(k+2-j)}{j!} \\
\frac{1}{\binom{m+1}{j}} &= \frac{j!}{(m+1)(m)(m-1)\cdots(m+2-j)}
\end{aligned}$$

Therefore, we have

$$\frac{\binom{m+1}{k}\binom{k+1}{j}}{\binom{m+1}{j}} = \frac{(m+1)(m)(m-1)\cdots(m+2-k)}{(m+1)(m)(m-1)\cdots(m+2-j)} \cdot \frac{(k+1)(k)(k-1)\cdots(k+2-j)}{k(k-1)(k-2)\cdots 1}$$

$$= (m+1-j)(m-j)(m-1-j)\cdots(m+2-k) \cdot \tfrac{(k+1)(k)(k-1)\cdots(k+2-j)}{k(k-1)(k-2)\cdots 1}$$

$$= (m+1-j)(m-j)(m-1-j)\cdots(m+2-k) \cdot \tfrac{k+1}{(k+1-j)(k-j)(k-1-j)\cdots 1}$$

Now, multiply by $\frac{m+2-j}{m+2-j}$ and rearrange...

$$\frac{\binom{m+1}{k}\binom{k+1}{j}}{\binom{m+1}{j}} = \frac{k+1}{m+2-j} \cdot \frac{(m+2-j)(m+1-j)\cdots(m+2-k)}{(k+1-j)(k-j)\cdots 1}$$

$$= \frac{k+1}{m+2-j} \cdot \frac{(m+2-j)(m+1-j)\cdots((m+2-j)-(k+1-j)+1)}{(k+1-j)!}$$

$$= \frac{k+1}{m+2-j} \cdot \binom{m+2-j}{k+1-j}$$

Notes for problem 30 on page 238. $(x+x^2+x^3+x^4+x^5+x^6)^2$ is the same as

$$(x+x^2+x^3+x^4+x^5+x^6)(x+x^2+x^3+x^4+x^5+x^6)$$

Now, imagine multiplying this out, and think about how you get the term x^5. One way is to multiply x from the first parenthesis and x^4 from the second parenthesis. You can also multiply x^3 from the first parenthesis and x^2 from the second parenthesis. And so on. The coefficient of x^5 will give the number of ways you can pick x^n from the first parenthesis and x^m from the second parenthesis such that $x^n x^m = x^{n+m} = x^5$. Thus, it gives the number of ways you can pick n from the first set and m from the second set such that $n + m = 5$.

Notes for problem 42 on page 239. The left-hand side of the equation gives the sum of the cardinalities of all the subsets of an n-element set. (See problem 40.)

Here's another way to count this sum. First, list every subset of an n-element set. For example, if $n = 3$, the following would be listed:

$$\{\ \}, \{A\}, \{B\}, \{C\}, \{A,B\}, \{A,C\}, \{B,C\}, \{A,B,C\}$$

Now, we just count the total number of letters above. We know that there are

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

subsets in total. And half of them contain the element A (Why?). Thus, there are

$$\frac{1}{2} \cdot 2^n = 2^{n-1}$$

subsets containing A. In other words, there are 2^{n-1} total A's in the list above. Likewise, there are 2^{n-1} total B's and C's. Thus, there are $n \cdot 2^{n-1}$ letters in total.

Notes for problem 49 **on page 255.** The student's recurrence is given by

$$P(n,k) = k \cdot P(n-1,k) + (k-1) \cdot P(n-1,k-1)$$

Using this formula, we generated the fifth row of this triangle.

$$1_{[1]} \quad 31_{[2]} \quad 180_{[3]} \quad 390_{[4]} \quad 360_{[5]} \quad 120_{[6]}$$

These numbers do in fact add to 1082, but what does each number represent? Well, here is the table from page 247:

k	$\left\langle \begin{matrix} 5 \\ k \end{matrix} \right\rangle$
0	0
1	1
2	30
3	150
4	240
5	120

It seems that

$$P(n,k) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$$

For example, the third number of the fifth row, namely 180, can be obtained by adding $\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle$ and $\left\langle \begin{matrix} 5 \\ 2 \end{matrix} \right\rangle$. This, along with

Theorem 4, explains why the sum of each row of this triangle gives the number of combinations in an n-button Simplex lock.

Finally, to show why

$$P(n,k) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$$

let

$$Q(n,k) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle$$

and show that $Q(n,k)$ satisfies the same recurrence as $P(n,k)$. This can be done fairly easily using Theorem 6.

Notes for problem 66a **on page 268.** Consider the combination

$$\{\{1,2\},\{4\},\{3,5\}\}$$

On a regular Simplex lock, there are 3! different combinations that use these three pushes. They are

$$\begin{gathered} \{\{1,2\},\{4\},\{3,5\}\} \\ \{\{1,2\},\{3,5\},\{4\}\} \\ \{\{4\},\{1,2\},\{3,5\}\} \\ \{\{4\},\{3,5\},\{1,2\}\} \\ \{\{3,5\},\{1,2\},\{4\}\} \\ \{\{3,5\},\{4\},\{1,2\}\} \end{gathered}$$

On this variant, however, these six combinations count as one.

In a similar manner, we can group together all simplex lock combinations that use the same three pushes into one. And each group contains 3! or six combinations. Thus, there are

$$\frac{\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle}{3!} = \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} = 25 \quad \text{combinations.}$$

By the way, this variant of the Simplex lock is equivalent to the scenario given in problem 8 on page 228. (Why?)

Notes for problem 66b **on page 268.** There are

$$\begin{aligned} &\left\{ \begin{matrix} 5 \\ 0 \end{matrix} \right\} + \left\{ \begin{matrix} 5 \\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} 5 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} + \left\{ \begin{matrix} 5 \\ 5 \end{matrix} \right\} \\ &= 0 + 1 + 15 + 25 + 10 + 1 \\ &= 52 \quad \text{combinations in total.} \end{aligned}$$

Notes for problem 66c **on page 268.** Does Theorem 4 also apply to this version of the Simplex lock?

Bibliography

[1] EDC. *The CME Project*, a comprehensive high school program funded by NSF grant ESI 02 42476, forthcoming from Prentice Hall, ≈ 2006.

[2] Niven, I., *The Mathematics of Choice*, MAA, Washington, DC, 1965.

[3] Velleman, D.J. and Gregory S. Call, "Permutations and Combination Locks", Mathematics Magazine Vol 68 No 4, October, 1995

[4] Harvey, B. Computer Science Logo Style, (3 vols), MIT Press, 1997.

[5] Petkovvsek, M., Herbert Wilf, and Doron Zeilberger, *A=B*, A. K. Peters Ltd., Wellesley, MA, 1996.

[6] Graham, R.L., D.E. Knuth, O. Patasknik, *Concrete Mathematics*, Addison-Wesley Publishing Company, 1992.

