1 Sets and Partitions

A *set* is a grouping of numbers (or people, or fruits, etc.), like $\{1, 2, 3\}$. A *subset* is a grouping of numbers (or people, or fruits, etc.) that may or may not contain any of the original set: $\{1, 3\}$ is a subset (and it's the same subset as $\{3, 1\}$). One subset contains all the elements, and one subset contains none of them: the notation, $\{\}$, is called the *empty set*.

1. (a) How many subsets of $\{1, 2, 3\}$ have exactly two elements?
   (b) How many subsets of $\{1, 2, 3\}$ are there?
   (c) How many subsets of $\{1, 2, 4, 8, 16\}$ have exactly three elements?

2. You can *partition* a set of numbers into non-empty subsets. For example, the set $\{1, 2, 3\}$ can be partitioned into two subsets: $\{1, 3\}$ and $\{2\}$ (which is the same as $\{2\}$ and $\{1, 3\}$). Or, it can be partitioned into two other subsets: $\{1, 2\}$ and $\{3\}$. It can even by partitioned into one or three subsets, thought not in particularly exciting ways.
   (a) How many total ways are there to partition $\{1, 2, 3\}$ into two subsets?
   (b) How many total ways are there to partition $\{1, 2, 3, 4\}$ into two subsets?
   (c) How many total ways are there to partition $\{1, 2, 3, 4, 5\}$ into two subsets?
   (d) How many total ways are there to partition $\{1, 2, 3, 4, 5, 6\}$ into two subsets?
   (e) What's going on?

3. Complete this table, with the number of elements as rows and the number of subsets as columns.

	1	2	3	4	5
1		–	–	–	–
2			–	–	–
3	1	3	1	–	–
4		7			–
5					

   Continue this table until you find a recursive rule you could use to continue the table even further.

4. Suppose we have the set

$$\{\text{Alicia, Bill, Claudia}\}.$$

(a) List all the possible ways to partition this set into exactly two non-empty subsets.

(b) List all the possible ways to partition this set into exactly one non-empty subset.

(c) Using your results from (a) and (b), derive all possible ways to partition the set

$$\{\text{Alicia, Bill, Claudia, Donna}\}$$

into exactly two non-empty subsets.

5. Consider again the set

$$\{\text{Alicia, Bill, Claudia}\}.$$

(a) List all the possible ways to partition this set into exactly three non-empty subsets.

(b) List all the possible ways to partition this set into exactly two non-empty subsets.

(c) Using your results from (a) and (b), derive all possible ways to partition the set

$$\{\text{Alicia, Bill, Claudia, Donna}\}$$

into exactly three non-empty subsets.

6. Let $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denote the number of ways to partition a set of n people into k non-empty subsets.

(a) Using Problem 4, explain why

$$\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 2 \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\}.$$

(b) Using Problem 5, explain why

$$\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} = 3 \left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\}.$$

(c) Use a "grouping" argument to explain why the following is true in general:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}.$$

7. Once again, complete the table:

	1	2	3	4	5	6
1		–	–	–	–	–
2			–	–	–	–
3	1	3	1	–	–	–
4		7			–	–
5						–
6						

8. On a 6-button Simplex Lock, find the number of combinations that use all six buttons and contains exactly 2 pushes. (Hint: It's *not* 31.)

9. Nick, Ellie, Megan, and Lynda are flying home from Park City. Each person can fly either first-class or coach. Thus, one possible seating arrangement is:

first-class	coach
Nick, Ellie	Megan, Linda

List all possible seating arrangements, provided that at least one person must fly first-class and at least one person must fly coach.

10. In Problem 9, we take the set

$$\{\text{Nick, Ellie, Megan, Lynda}\}$$

and *order-partition* it into two subsets

$$\{\text{Nick, Ellie}\} \text{ and } \{\text{Megan, Lynda}\}$$

which we consider to be *different* from

$$\{\text{Megan, Lynda}\} \text{ and } \{\text{Nick, Ellie}\}.$$

(a) How many total ways are there to order-partition a set of three people into exactly two non-empty subsets?

(b) How many total ways are there to order-partition a set of four people into exactly two non-empty subsets?

(c) How many total ways are there to order-partition a set of five people into exactly two non-empty subsets?

11. Complete this table, with the number of elements (people, numbers, etc.) as rows and the number of non-empty subsets as columns.

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	1	–	–	–	–
2	–			–	–	–
3	–		6		–	–
4	–					–
5	–					

12. Continue the table from Problem 11 until you find a recursive rule that you could use to continue the table even further.

13. How are tables in Problems 7 and 11 related?

14. The newly created American Dodgeball League has five teams which must be placed into three divisions. The teams are: Giants, Eagles, Rams, Panthers, and Jets. The divisions are: East, Central, and West. For example, the teams can be divided as follows.

East	Central	West
Giants, Panthers	Eagles	Rams, Jets

Each division must contain at least one team. In how many different ways can the teams be divided?

15. A class has five people:

$$\{\text{Alicia, Bill, Claudia, Donna, Eugene}\}.$$

Their teacher breaks them up into the following three groups:

- Alicia, Eugene
- Bill
- Claudia, Donna

Each group must give a presentation on one of the following topics: Algebra, Geometry, and Combinatorics. And no two groups can pick the same topic. In how many different ways can the three topics be assigned to the three groups?

16. Recall that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ denotes the number of ways to partition a set of n people into k non-empty subsets.

(a) Find the value of $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$.

(b) Now let $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ denote the number of ways to *order-partition* a set of n people into k non-empty subsets. (E.g. we count {Nick, Ellie} and {Megan, Lynda} to be *different* from {Megan, Lynda} and {Nick, Ellie}.) Find the value of $\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle$.

(c) How are $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$ and $\left\langle \begin{matrix} 5 \\ 3 \end{matrix} \right\rangle$ related?

(d) How are $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ and $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ related?

17. We have seen the following recursive rule for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\},$$

which led to the following table (the number of elements are the rows and the number of non-empty subsets are the columns):

	1	2	3	4	5
1	1	–	–	–	–
2	1	1	–	–	–
3	1	3	1	–	–
4	1	7	6	1	–
5	1	15	25	10	1

Now complete the following table for $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$:

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	1	–	–	–	–
2	–			–	–	–
3	–		6		–	–
4	–					–
5	–		30			

18. Consider the set

$$\{\text{Alicia, Bill, Claudia}\}.$$

(a) List all possible ways to *order-partition* these people into exactly two non-empty subsets.

(b) List all possible ways to order-partition them into exactly one non-empty subset.

(c) Now Donna shows up and wants to join in the fun. List all the possible ways to order-partition

$$\{\text{Alicia, Bill, Claudia, Donna}\}$$

into exactly two non-empty subsets.

19. Again, consider the following people

$$\{\text{Alicia, Bill, Claudia}\}.$$

(a) List all possible ways to *order-partition* these people into exactly three non-empty subsets.

(b) List all possible ways to order-partition them into exactly two non-empty subsets.

(c) Using your results from (a) and (b), derive all possible ways to order-partition the set

$$\{\text{Alicia, Bill, Claudia, Donna}\}$$

into exactly three non-empty subsets.

20. (a) Using Problem 18, explain why

$$\left\langle \begin{array}{c} 4 \\ 2 \end{array} \right\rangle = 2 \left\langle \begin{array}{c} 3 \\ 2 \end{array} \right\rangle + 2 \left\langle \begin{array}{c} 3 \\ 1 \end{array} \right\rangle.$$

(b) Using Problem 19, explain why

$$\left\langle \begin{array}{c} 4 \\ 3 \end{array} \right\rangle = 3 \left\langle \begin{array}{c} 3 \\ 3 \end{array} \right\rangle + 3 \left\langle \begin{array}{c} 3 \\ 2 \end{array} \right\rangle.$$

(c) Explain why

$$\left\langle \begin{array}{c} n \\ k \end{array} \right\rangle = k \left\langle \begin{array}{c} n-1 \\ k \end{array} \right\rangle + k \left\langle \begin{array}{c} n-1 \\ k-1 \end{array} \right\rangle.$$

2 Breaking the Lock

1. (a) On a 5-button Simplex Lock, find the number of combinations that use all five buttons and contain exactly three pushes.

   (b) On a 5-button Simplex Lock, find the number of combinations that use all five buttons.

2. Let $T(n)$ be the total number of combinations on an n-button Simplex Lock, and $L(n)$ the number of combinations on an n-button Simplex Lock that use all n buttons. Fill in the table below.

n	$L(n)$	$T(n)$
1		
2		
3		
4		150
5		1082

   Any thoughts?

3. (a) On a 3-button Simplex Lock, list all combinations that use all three buttons.

   (b) On a 3-button Simplex Lock, list all combinations that do *not* use all three buttons.

   (c) Can you find a *mapping* between the lists in (a) and (b) that explains why there are just as many of each?

4. Use Problem 3 to explain why the total number of combinations on a Simplex Lock is exactly twice the number of combinations that use all the buttons (i.e. $T(n) = 2L(n)$).

3 The Mahler Basis

In this section, we will discuss an interesting connection between partitioning of sets (from Section 1) and a certain family of polynomials. For an integer $k \geq 1$, the kth *combinatorial polynomial*, $\binom{x}{k}$, is defined by

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

For example,

$$\begin{aligned}\binom{x}{4} &= \frac{x(x-1)(x-2)(x-3)}{4!} \\ &= \frac{x^4 - 6x^3 + 11x^2 - 6x}{24} \\ &= \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{11}{24}x^2 - \frac{1}{4}x.\end{aligned}$$

By convention, we also put $\binom{x}{0} = 1$.

1. Let
$$f(x) = 3\binom{x}{5} - 2\binom{x}{4} + 5\binom{x}{3} - 4\binom{x}{1} + 7\binom{x}{0}.$$
Write f as a linear combination of the powers of x.

2. Show that
$$x^4 = 24\binom{x}{4} + 36\binom{x}{3} + 14\binom{x}{2} + \binom{x}{1}.$$

3. Let
$$g(x) = 3x^5 - 2x^4 + 5x^3 - 4x + 7.$$
Write g as a linear combination of the $\binom{x}{k}$.

Of course, the "normal" way to express a polynomial is as a linear combination of powers of x, e.g.

$$f(x) = \frac{x^5}{40} - \frac{x^4}{3} + \frac{53x^3}{24} - \frac{14x^2}{3} - \frac{37x}{30} + 7. \qquad (*)$$

For some purposes (Newton's difference formula, for example), it turns out to be more useful to express f in terms of the $\binom{x}{k}$, i.e. in terms of the *Mahler basis*:

$$f(x) = 3\binom{x}{5} - 2\binom{x}{4} + 5\binom{x}{3} - 4\binom{x}{1} + 7\binom{x}{0}. \qquad (**)$$

(You showed in Problem 1 that $(*)$ and $(**)$ give the same polynomial.) Thus, it will be useful to be able to convert back and forth between "normal" and Mahler bases. One direction is fairly straightforward.

4. Define the numbers $b_{n,k}$ by
$$\binom{x}{n} = \sum_{k=0}^{n} b_{n,k} \cdot x^k.$$
For example, since
$$\binom{x}{4} = -\frac{1}{4}x + \frac{11}{24}x^2 - \frac{1}{4}x^3 + \frac{1}{24}x^4,$$

we have

$$b_{4,0} = 0$$
$$b_{4,1} = -1/4$$
$$b_{4,2} = 11/24$$
$$b_{4,3} = -1/4$$
$$b_{4,4} = 1/24$$

and $b_{4,k} = 0$ for $k > 4$.

Fill in the following table containing the values $b_{n,k}$, with n as rows and k as columns. What patterns can you find?

	0	1	2	3	4	5
0	1					
1						
2						
3						
4	0	$-1/4$	$11/24$	$-1/4$	$1/24$	0
5						

5. Define the numbers $a_{n,k}$ by

$$x^n = \sum_{k=0}^{n} a_{n,k} \binom{x}{k}.$$

For example, we saw in Problem 2 that

$$x^4 = 0\binom{x}{0} + 1\binom{x}{1} + 14\binom{x}{2} + 36\binom{x}{3} + 24\binom{x}{4}$$

so that

$$a_{4,0} = 0$$
$$a_{4,1} = 1$$
$$a_{4,2} = 14$$
$$a_{4,3} = 36$$
$$a_{4,4} = 24$$

and $a_{4,k} = 0$ for $k > 4$.

Fill in the following table containing the values $a_{n,k}$, with n as rows and k as columns.

	0	1	2	3	4	5
0	1					
1						
2						
3						
4	0	1	14	36	24	0
5						

6. Compare the tables of the $\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ from Section 1 and of the $a_{n,k}$. What's going on here?